

Asymptotic Expansions of Determinants for Families of Trace Class Operators

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I. Introduction. Suppose X is a trace class operator on Hilbert space. Then e^X differs from the identity by a trace class operator and so its determinant is defined [2, Chapter IV]. In fact

$$(1.1) \quad \det e^X = e^{\text{tr} X}.$$

Now take a projection operator P . Then Pe^XP , thought of as acting on the range of P , also differs from the identity by a trace class operator and so its determinant is defined. What is it?

We shall describe a procedure which produces an asymptotic expansion for this determinant in case X is a family of operators depending on a parameter in a particular way. The context will be general enough to include the continuous analogue of the Szegő limit theorem for Toeplitz determinants with extensions to variable convolutions, higher dimensions, and general Riemannian manifolds. A particular case is easy to state and we do so here.

Let Ω be a bounded subset of R^d with C^∞ boundary and $k(z)$ a function on R^d whose Fourier transform

$$\sigma(\xi) = \int e^{-i\xi \cdot z} k(z) dz$$

satisfies

(i) for each multi-index k ,

$$\frac{\partial^k \sigma}{\partial \xi^k} = ((1 + |\xi|)^\omega)^{-|k|},$$

where $\omega < -d$;

(ii) there is a continuous determination of

$$\log(1 + \sigma(\xi))$$

which tends to 0 as $|\xi| \rightarrow \infty$.

Then the determinant of the operator on $L_2(\Omega)$