

Commutative Subspace Lattices

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1. Introduction. In this paper, we will show that if \mathfrak{A} is a reflexive algebra with a commutative subspace lattice, then the “essentially invariant” projections of \mathfrak{A} are just the compact perturbations of the invariant projections. We will also do a detailed analysis of algebras with finite commutative lattices which leads to certain distance estimates and a simplified proof of Arveson’s theorem that commutative lattices are reflexive.

The interest of this study lies in the fact that the algebras considered form a large class of non self-adjoint operators algebras. These algebras were analyzed extensively by Arveson in [2] where he developed some difficult machinery to handle the problem. The approach used here is more elementary, and it is hoped that this will help clarify the situation somewhat.

Let \mathfrak{A} be an algebra of operators on a Hilbert space \mathcal{H} . Let \mathfrak{L} be the lattice of invariant subspaces of \mathfrak{A} . If \mathcal{L} is a lattice of subspaces, then $\text{alg } \mathcal{L}$ is the algebra of all operators leaving \mathcal{L} invariant. \mathfrak{A} is reflexive if $\mathfrak{A} = \text{alg lat } \mathfrak{A}$. If P is a projection, we will denote $I - P$ by P^\perp . Let π be the canonical homomorphism of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra. Let $\mathcal{L}_e = \text{lat } \pi\mathfrak{A}$ be the lattice of *essentially invariant* projections of \mathfrak{A} . Since any projection in the Calkin algebra can be lifted to a projection P in $\mathcal{B}(\mathcal{H})$, we consider those projections P such that $P\mathfrak{A}P$ is compact for every A in \mathfrak{A} .

A remarkable theorem of Voiculescu [4] states that if \mathfrak{A} is a norm closed separable subalgebra of the Calkin algebra, then it is reflexive. In particular, separable subalgebras of the Calkin algebra have many essentially invariant subspaces. We are interested in the question of when the essentially invariant projections correspond to a lattice of *bona fide* invariant subspaces of \mathcal{H} . In particular, we ask: what conditions on a reflexive algebra ensure that $\mathcal{L}_e = \pi\mathcal{L}$? The condition that the invariant projections of a lattice commute is a “smallness” condition that in some sense corresponds to separability for algebras. In this sense, the following theorem is dual to Voiculescu’s theorem. This result was part of the author’s Ph.D. dissertation.

Theorem 1.1. *If \mathcal{L} is a commutative lattice and $\mathfrak{A} = \text{alg } \mathcal{L}$ then $\mathcal{L}_e = \pi\mathcal{L}$.*

In [1], Arveson analyzes operator algebras with a linearly ordered lattice. He points out that for any algebra \mathfrak{A} with lattice \mathcal{L} and any operator T in $\mathcal{B}(\mathcal{H})$,