

The Mackey and Strong Topologies on Köthe Sequence Spaces

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In this article we derive some new properties of Köthe sequence spaces. The main result is that for any Köthe sequence space S , $c_0S = \{us : u \in c_0, s \in S\}$ is precisely equal to the closure of ϕ in S with respect to the strong S^α -topology. This main result is a far reaching generalization of the problem in elementary analysis to show, that if $\sum_n a_n$ converges absolutely then $a_n = b_n c_n$ where (b_n) converges to 0 and $\sum_n c_n$ converges absolutely. A special case of the main result was noted by Garling (Corollary 3, p. 1007 of [2]) when S has an FK -topology. The central theorem enables us to give a unified discussion of the relation of the Mackey and strong topology on a Köthe space as well as questions of semi-reflexivity and reflexivity thus supplementing the discussion in §30 of [4]. A more surprising consequence of the main result is Corollary 2.4 which asserts that the S^α -normal topology on a Köthe spaces S is a Schwartz topology if and only if S^α -bounded subsets of S are S^α -precompact. We have also included a discussion of the inclusion of m and ℓ^1 in perfect spaces which generalizes the work of James for unconditional bases in Banach spaces and Sargent for BK -spaces; these generalizations have possibly been anticipated although to our knowledge not explicitly set out.

The discussion will be based on §30 of [4], and we will use other basic ideas found in this book. The n^{th} coordinate of a sequence t will be denoted by $t(n)$; $t[\cong k]$ denotes the sequence s for which $s(n) = 0$ for $n < k$ and $s(n) = t(n)$ for $n \geq k$; $t[\leq k]$ has an analogous meaning. The sequence with 1 in the i^{th} place and 0's elsewhere is denoted e_i . The product st of two sequences is defined coordinatewise; the product AB of two sets of sequences A and B consists of all sequences st as s ranges over A and t over B . The space of all sequences is denoted by ω ; the space of all sequences which have only finitely many non-zero coordinates by ϕ ; the space of bounded sequences, by m ; the space of zero-convergent sequences ℓ^1 . For S any sequence space, S^α the Köthe dual of S denotes the space of all t such that $st \in \ell^1$ for each s in S . A sequence space S is said to be normal if $mS \subset S$. A locally convex space S of sequences is said to have AD if ϕ is dense in S ; and is said to have AK if $s[\leq n]$ converges to s for all s in S . The space S is said to be a Köthe space if $S^{\alpha\alpha} = S$.