

On the Existence of Radial Boundary Values for Functions Subharmonic in a Lipschitz Domain

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1. Introduction. A classical theorem of Littlewood [7] states that if u is subharmonic in the unit disc and has a non-negative harmonic majorant, then u has radial limits a.e. For generalizations of this result we refer to Doob [3], Widman [11] and the references quoted in these papers. The purpose of this note is to study the analogue of Littlewood's theorem for functions subharmonic in a Lipschitz domain $D \subset \mathbb{R}^n$. More precisely we want to find conditions on a function $e: \partial D \rightarrow S^{n-1}$, where S^{n-1} is the unit sphere in \mathbb{R}^n , such that $\lim_{t \downarrow 0} u(P + te(P))$ exists a.e. with respect to the surface measure of ∂D .

Let $E \subset \partial D$ and let $e: E \rightarrow S^{n-1}$ be a function. We say that e is *nontangential* if to each point $P \in E$ there is an open truncated circular cone $\Gamma(P)$ with vertex at P and axis of symmetry $e(P)$ such that $\Gamma(P) \subset D$.

Theorem 1. *Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain and let $E \subset \partial D$ be Borel measurable. Suppose that $e: E \rightarrow S^{n-1}$ is nontangential and that there is a constant M such that*

$$(1.1) \quad |e(P) - e(Q)| \leq M|P - Q|, \quad P, Q \in E.$$

If u is subharmonic in D and has a non-negative harmonic majorant then $\lim_{t \downarrow 0} u(P + te(P))$ exists and is finite a.e. on E .

We would like to point out that condition (1.1) is sharp in the sense that it cannot in general be replaced by a Hölder condition of exponent less than one as the following theorem shows.

Theorem 2. *Given α , $0 < \alpha < 1$, there is a $C^{1,\alpha}$ -domain $D \subset \mathbb{R}^2$ with the following property: There is a compact set $F \subset \partial D$ of positive length and a negative subharmonic function u in D such that $\liminf_{t \downarrow 0} u(P + tN(P)) = -\infty$ for all $P \in F$, where $N(P)$ denotes the unit inward normal to ∂D at P . In addition, N is nontangential and $|N(P) - N(Q)| \leq M|P - Q|^\alpha$.*

We remark that it also follows from Theorem 2 that even for $C^{1,\alpha}$ -domains one does not in general have convergence along the normals.