

Sets and Functions of Finite L^p -Capacity

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1. Introduction. If C is a measure on the measure space X and φ a complex valued measurable function on X , then it is well known that φ is p -th power summable over X with respect to C iff $J(\varphi)^p \equiv \int_0^\infty C(\{x \in X: |\varphi(x)| > t\}) dt^p < \infty$.

This integral, however, is well defined for a much more extensive class of set functions C . In particular, the purpose of this paper is to study J when C is an L^p -Bessel capacity $B_{m,p}$ (or L^p -Riesz capacity $R_{m,p}$) on the algebra of all subsets of $X = R^n$. Here m is a positive integer. We give two interesting applications of J in this case: the study of the growth of Sobolev functions, on bounded or unbounded thin sets, by establishing an extension theorem for Sobolev functions (Section 4), and a characterization of sets of finite Riesz capacity (Section 7).

These results are made possible by first showing that $J(\varphi)$ is comparable to a sublinear L^p -capacitary functional (Section 3). This functional has the important property that $J(\varphi) < \infty$ iff there is a Bessel (or Riesz) potential of a non-negative L^p function that dominates $|\varphi|$ quasi-everywhere on R^n . Furthermore, this functional reduces to Bessel (or Riesz) capacity when φ is the indicator function of a set. Consequently, we naturally think of a φ with $J(\varphi) < \infty$ as a function of finite capacity.

The extension theorem of Section 4 is an outgrowth of the following example. Suppose Ω is an open set of R^n with finite n -measure. Then $\varphi \equiv 1$ restricted to Ω is clearly in $W^{1,p}(\Omega)$. So the question is: when can this function be extended to $W^{1,p}(R^n)$, *i.e.* when is there a function $\Phi \in W^{1,p}(R^n)$ such that $\Phi = \varphi$ on Ω ? Or, in other words, on what kind of sets Ω can a Sobolev function Φ be identically one? The necessary and sufficient condition for this to happen is well known, namely Ω must have finite Bessel capacity, $B_{1,p}(\Omega) < \infty$. (When $p = 2$, Φ is often referred to as the equilibrium potential for the variational problem: $\min \int (|\nabla u|^2 + u^2) dx$, $u \in W^{1,2}(R^n)$ and $u = 1$ on Ω .)

More generally, suppose that φ is given on R^n and that it belongs to $W^{m,p}(\Omega)$ but not to $W^{m,p}(R^n)$. The necessary and sufficient condition (Theorem 4.2) on Ω , now becomes $\int_0^\infty B_{m,p}(\Omega \cap \{|\varphi| > t\}) dt^p < \infty$, provided the derivatives of φ satis-