

On Commuting Self-Adjoint Extensions of Unbounded Operators

S. P. SLINKER

1. Introduction. If A and B are symmetric operators defined on a dense domain \mathcal{D} in a separable Hilbert space \mathcal{H} which satisfy $A\mathcal{D} \subset \mathcal{D}$, $B\mathcal{D} \subset \mathcal{D}$, and $AB\psi = BA\psi$ for all $\psi \in \mathcal{D}$, we find sufficient conditions that A and B have commuting self-adjoint extensions; namely that A and B are real with respect to a conjugation J , and A has a self-adjoint extension A_0 , also real with respect to J , with the property that the unitary group generated by A_0 commutes weakly (Definition 2.2) with B . This result is applied to the case where \mathcal{D} is a set of analytic vectors for A . In the last section the connection between certain positivity conditions and the existence of commuting self-adjoint extensions is studied.

2. Commutativity. The commutativity of unbounded operators poses difficulties because of the problems with the domains involved. We are concerned with the case where the unbounded operators are symmetric or self-adjoint. We first consider commutativity when one of the operators is bounded. Suppose A is a symmetric operator, i.e., $A \subset A^*$, defined on the dense domain $\mathcal{D}(A)$ in the separable Hilbert space \mathcal{H} . Let B be an element of $\mathcal{B}(\mathcal{H})$, the bounded operators on \mathcal{H} .

Definition 2.1. B commutes strongly with A provided $AB \supset BA$. That is, if $\psi \in \mathcal{D}(A)$ then $B\psi \in \mathcal{D}(A)$ and $AB\psi = BA\psi$.

If B commutes strongly with A then it commutes strongly with the closure \bar{A} of A . If A is self-adjoint, i.e., $A = A^*$, then B commutes strongly with A if and only if it commutes with the spectral projections of A in the usual sense [11, Theorem 8.1].

A weaker notion of commutativity is the following [1];

Definition 2.2. If A is a symmetric operator on the dense domain $\mathcal{D}(A)$ and $B \in \mathcal{B}(\mathcal{H})$ then B commutes weakly with A provided $(\psi, BA\phi) = (A\psi, B\phi)$ for all $\psi, \phi \in \mathcal{D}(A)$.