

# Ratio Limit Theorems for Hitting Times and Hitting Places

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**1. Introduction and statement of results.** Let  $\{X_n\}$  be a recurrent random walk generated by a probability measure  $\mu$ , with state space  $S \subseteq \mathbb{R}^d (d \leq 2)$ . For any Borel set  $B \subseteq S$  let  $T_B = \inf \{n > 0: X_n \in B\}$  ( $=\infty$  if no such  $n$  exists) be the hitting time of  $B$  and set  $R_n = P^D(T_D > n) \equiv \int_D P^x(T_D > n) dx$  where  $D$  is the unit ball in  $S$ . Here  $P^x(\cdot)$  denotes  $P(\cdot | X_0 = x)$  and  $dx$  denotes Haar measure ( $S$  is assumed equal to  $\mathbb{R}^d$ ,  $\mathbb{Z}^d$  or  $\mathbb{R} \times \mathbb{Z}$ ).

In [2] Kesten and Spitzer obtained the following ratio limit theorem for  $S = \mathbb{Z}^d$ :

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{P^x(T_B > n)}{R_n} = L_B(x) < \infty$$

for any  $x \in S$  and any bounded set  $B \subseteq S$ . They also showed that for  $A \subseteq B$

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{P^x(T_B > n; X_{T_B} \in A)}{R_n} = L_B(x) \int_A \hat{L}_B(y) dy < \infty,$$

where  $\hat{L}_B$  is  $L_B$  defined for the reversed random walk (generated by the measure  $\hat{\mu}$  defined by  $\hat{\mu}(A) = \mu(-A)$ ,  $A \subseteq \mathbb{R}^d$ ). An intuitive interpretation of (1.2) when  $L_B(x)$  is nonzero is that if a long time has elapsed and the process has not yet hit  $B$ , then its hitting place in  $B$  does not depend on its starting point. For  $S = \mathbb{Z}^d$  one may also write (1.2) as

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{E^x[g(X_{T_B}); T_B > n]}{R_n} = L_B(x) \int_B g(y) \hat{L}_B(y) dy < \infty$$

for any real-valued function  $g$  on  $B$ . This formulation is of interest due to potential theoretic considerations. Namely, let the operator  $\Delta$  on bounded Borel measurable function on  $S$  be defined by

$$\Delta f(x) = \int P^x(X_1 \in dy) f(y) - f(x).$$