

Convolutions of Starlike Functions

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1. Introduction. Let $\Delta = \{z : |z| < 1\}$, $\bar{\Delta} = \{z : |z| \leq 1\}$, and $T = \{z : |z| = 1\}$. Given α , $-\infty < \alpha \leq 1$, let $S(\alpha)$ be the class of normalized starlike functions of order α in Δ . An analytic function f in Δ is in $S(\alpha)$ if and only if $f(0) = 0, f'(0) = 1$, and

$$(1.1) \quad \operatorname{Re}[zf'(z)/f(z)] \geq \alpha, \quad z \in \Delta.$$

We shall say that an analytic function g on $\bar{\Delta}$ is starlike of order α on $\bar{\Delta}$, if for some $R > 1$, the function $z \rightarrow g(Rz)$, is a constant multiple of an $f \in S(\alpha)$. Using the Poisson integral formula and integrating (1.1), it follows that $f \in S(\alpha)$ if and only if

$$(1.2) \quad \log (f(z)/z) = -2(1 - \alpha) \int_T \log (1 - e^{-i\theta}z) d\mu(e^{i\theta}), \quad z \in \Delta,$$

for some probability measure μ on T . Note that the function

$$(1.3) \quad z/(1 - z)^{2(1 - \alpha)} = \sum_{n=1}^{\infty} A_n(\alpha)z^n,$$

$$A_n(\alpha) = \Gamma(n + 1 - 2\alpha)/[\Gamma(2 - 2\alpha)(n - 1)!],$$

is in $S(\alpha)$, as follows easily from (1.2). Let $C(\alpha)$ denote the class of analytic functions g in Δ satisfying $g(0) = 0, g'(0) = 1$ and the condition that

$$(1.4) \quad \operatorname{Re}[zg'(z)/h(e^{i\theta}z)] \geq 0, \quad z \in \Delta,$$

for some $h \in S(\alpha)$ and θ real.

For fixed α , $-\infty < \alpha \leq 1$, and f, g analytic in Δ with $f(0) = g(0) = 0$, define the convolution of f and g [denoted $(f * g)_\alpha$] as follows: If

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

and

$$g(z) = \sum_{n=1}^{\infty} b_n z^n,$$