

Nonsingular Bilinear Forms, Generalized J Homomorphisms, and the Homotopy of Spheres I.

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A bilinear form

$$f : \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^c$$

is said to be *non-singular* iff

$$f(x, y) = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0.$$

For such a form f we define

$$f^+ : \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R} \oplus \mathbb{R}^c$$

by

$$f^+(x, y) = (\|x\|^2 - \|y\|^2, f(x, y)).$$

Then

$$f^+(x, y) = 0 \Leftrightarrow x = 0 \quad \text{and} \quad y = 0$$

so f^+ induces a map (the *Hopf construction*)

$$h(f) : S^{a+b-1} \rightarrow S^c$$

by

$$h(f)(x, y) = \frac{f^+(x, y)}{\|f^+(x, y)\|}.$$

The homotopy class of $h(f)$ represents an element of $\pi_{a+b-1}(S^c)$. Elements constructed in this manner have played a central role in the study of the homotopy groups of spheres. For example, by taking for f the multiplication of the (skew) fields \mathbb{C} , \mathbb{H} and \mathbb{K} , Hopf [8] constructed non-zero elements in $\pi_3(S^2)$, $\pi_7(S^4)$ and $\pi_{15}(S^8)$.

The general problem that concerns us is the representation *via* the Hopf construction of elements in the homotopy of spheres by bilinear forms.