

P. A. Smith-theoretic Algebro-geometric Invariants of Compact Differentiable Manifolds

R. S. KULKARNI

§1. The notion of algebraic deficiency. Let M^n be a compact differentiable manifold. By a theorem of Nash which extended a theorem of Seifert and which was completed by Tognoli (cf. [N], [S] and [T]), there exists an algebraic variety \mathbb{X} defined over \mathbb{R} such that $M = \mathbb{X}_{\mathbb{R}}$. The original proof gives an \mathbb{X} which is affine. But it may be shown further that one may choose \mathbb{X} either affine or projective which is moreover nonsingular and also $\mathbb{X}_{\mathbb{C}}$ can be embedded in \mathbb{C}_{2n+1} or $\mathbb{P}_{2n+1}(\mathbb{C})$ so that $\mathbb{X}_{\mathbb{C}}$ is invariant under the canonical conjugation $z \xrightarrow{\sigma} \bar{z}$ where z stands for an affine or projective co-ordinate system and σ restricted to $\mathbb{X}_{\mathbb{C}}$ coincides with the canonical conjugation defined by the \mathbb{R} -variety structure of \mathbb{X} cf. [K]. A nonsingular $\mathbb{X}_{\mathbb{C}}$ will be called an *affine* respectively *projective complexification* of M according to whether \mathbb{X} is affine or projective. The choice of \mathbb{X} is not unique so it is natural to ask for \mathbb{X} which is “minimal” in some sense. One such notion suitable for the affine case was investigated in [K]. Here I shall consider a notion which is suitable for the projective case. It is motivated by the P. A. Smith theory in the case of involutions.

We need only the following special case of the P. A. Smith theory. Let X be a finite complex with involution τ , i.e. a homeomorphism $\tau : X \rightarrow X$ with $\tau^2 = 1$. Let M be the fixed point set of τ . Let $H^*(A)$ denote the total cohomology module of a space A with \mathbb{Z}_2 -coefficients and $h^*(A)$ the dimension of $H^*(A)$ over \mathbb{Z}_2 . The Smith theory asserts that $h^*(X) - h^*(M)$ is an even nonnegative integer (cf. Bredon [B], Chapter 3 §4).

We apply this to the following situation. Let $X_n(1)$ be a complex manifold, $\tau : X \rightarrow X$ a conjugation, i.e. an antiholomorphic involution, and $M =$ the fixed point set of τ . Then M is either empty or an n -dimensional differentiable manifold. If M is nonempty we call X a *complexification* of M . (The previously defined affine or projective complexification of a differentiable manifold is clearly a complexification in the new sense.)