

On Some Existence Theorems for Semi-linear Elliptic Equations

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Introduction. In a recent paper [3] one of the authors showed that the semi-linear elliptic boundary-value problem (BVP)

$$(1) \quad \begin{cases} Au = f(x, u, Du) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

has one or several solutions provided that suitable sub- and supersolutions are known. Here A is a second order strongly elliptic differential operator on the bounded domain $\Omega \subseteq \mathbb{R}^N$, and B is a first order boundary operator of the sort which allows use of the maximum principle. Moreover, Du denotes the gradient of u and $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is required to satisfy certain assumptions detailed in Section 1. In this paper we obtain (slight) generalizations of these results by an argument which is considerably simpler and more direct than the presentation in [3].

The device used in [3] was to study the corresponding parabolic initial-boundary value problem

$$(2) \quad \begin{cases} \frac{\partial u}{\partial t} + Au = f(x, u, Du) & \text{in } \Omega \times (0, \infty) \\ Bu = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0 & \text{on } \bar{\Omega}. \end{cases}$$

Roughly, it was shown that solving (2) yields an order preserving compact semigroup $\{S(t) : t \geq 0\}$ on the order interval $[\bar{u}, \hat{u}]$ whenever \bar{u}, \hat{u} are ordered upper and lower solutions of (2). Rest points of $S(t)$ are solutions of (1), and one can obtain rest points of $S(t)$ by applying abstract results to $S(2^{-n})$ and letting $n \rightarrow \infty$. An essential point in this program is to associate solutions of (1) with fixed points of order preserving compact mappings with the same sub- and supersolutions as (1). A simpler idea which achieves this same end is the following: Let $\lambda > 0$, $h \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and solve