

# Almost Chebyshev Subsets in Reflexive Banach Spaces

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**§1.** Let  $K$  be a nonempty subset in a (real) Banach space  $X$ . For each  $x \in X$ , we say that  $y \in K$  is a *best approximation* from  $x$  to  $K$  if  $\|x - y\| = \inf \{\|x - z\| : z \in K\}$ . A set  $K$  is called *proximal (Chebyshev)* if every point  $x \in X$  admits a (unique) best approximation from  $K$ . It is easy to see that in a reflexive Banach space every weakly closed subset is proximal; however, the statement is not true for arbitrary (norm) closed sets. In [8] Stečkin proved that *for any closed subset  $K$  in a uniformly convex space, the set of points in  $X$  which fail to have unique best approximation from  $K$  is a set of first category*. We call a set with the above property *almost Chebyshev*. This concept has been studied by many authors (cf. e.g. [1], [2], [4], [5], [6]). A remaining unsolved problem is: Is every closed subset in a reflexive locally uniformly convex space almost Chebyshev [7, p. 375]? In this note, we give a positive answer. We remark that the answer is not true for reflexive strictly convex spaces [Edelstein, 2] or locally uniformly convex spaces [Cobzas, 1].

**§2.** Let  $X$  be a Banach space. We use  $X^*$  to denote the dual of  $X$ . A real-valued function  $f$  on  $X$  is said to be *Fréchet differentiable* if for each  $x \in X$ , there exists an  $x^* \in X^*$  such that for any  $\epsilon > 0$ , there exists  $\eta > 0$  which satisfies

$$|f(y) - f(x) - \langle x^*, y - x \rangle| \leq \epsilon \|y - x\| \quad \forall \|y - x\| < \eta.$$

For  $x \in X$ , a functional  $x^*$  in  $X^*$  is called a *local  $\epsilon$ -support* of  $f$  at  $x$  if there exists an  $\eta > 0$  such that

$$\|x - y\| < \eta \Rightarrow f(y) - f(x) \geq \langle x^*, y - x \rangle - \epsilon \|y - x\|.$$

Note that the  $x^*$  in the above definition depends on  $\epsilon$ . In [3], Ekeland and Lebourg proved

**Proposition 1.** *Suppose  $X$  is a Banach space which admits a nonnegative Fréchet differentiable function with bounded nonempty support. Then for any  $\epsilon > 0$  and for any lower semicontinuous function  $f$  on  $X$ , there exists a dense set of points in  $X$  where  $f$  is locally  $\epsilon$ -supported.*