

Some Sharp Inequalities for Conjugate Functions

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1. Introduction. We shall denote the unit disk $|z| < 1$ by D and the unit circle $|z| = 1$ by T . For a subset $E \subset T$, $|E|$ denotes the Lebesgue measure of E , and, for functions f on T , we write

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

Suppose that f is a real-valued integrable function on T . Associated with f is its *conjugate function* \tilde{f} , which may be defined a.e. on T as follows. Let $F(z)$ be the function analytic in D whose real part is the Poisson integral of f , with $\text{Im } F(0) = 0$. Thus, for $|z| < 1$,

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \frac{1 + ze^{-i\varphi}}{1 - ze^{-i\varphi}} d\varphi.$$

Then F has non-tangential limits $F(e^{i\theta})$ a.e. on T , and we have

$$\tilde{f}(e^{i\theta}) = \text{Im } F(e^{i\theta}).$$

An account of the theory of conjugate functions may be found in [12].

A famous theorem of M. Riesz asserts that if $1 < p < \infty$ and $f \in L^p(T)$ then also $\tilde{f} \in L^p(T)$ and, moreover,

$$\|\tilde{f}\|_p \leq C_p \|f\|_p.$$

Here, and throughout the paper, C_p denotes a constant which depends only on p , while C will denote an absolute constant. Also, we will always assume that f is real valued. Riesz's theorem fails for $p = 1$. Indeed, there exist functions $f \in L^1(T)$ for which $\tilde{f} \notin L^1(T)$. However, in this case another remarkable inequality, what is now called the "weak 1-1 inequality", holds and was proved a few years earlier by Kolmogorov [8]. For $t \geq 0$ define the set $E_t(f) \subset T$ by

$$E_t(f) = \{z \in T : |f(z)| \geq t\}.$$

Weak 1-1 inequality. For every $t > 0$,

$$(1) \quad |E_t(\tilde{f})| \leq C t^{-1} \|f\|_1.$$