

The Maximal Nonnegative Reducing Subspaces of J -selfadjoint Operators

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Introduction. R. S. Phillips has conjectured that every commutative family of J -unitary (or J -selfadjoint) operators on a Krein space \mathbf{K} with indefinite inner product J has a maximal nonnegative invariant subspace, and he has shown that a normal, J -unitary or J -selfadjoint operator N has a reducing maximal nonnegative subspace as required by the conjecture. Using generalized inverses, we construct maximal nonnegative reducing subspaces for such an operator N , and we characterize the set of J -selfadjoint operators having maximal nonnegative reducing subspaces. Also, we demonstrate that a normal J -selfadjoint (or J -unitary) operator cannot have a uniformly positive maximal nonnegative invariant subspace, except in the trivial case of an operator which is also selfadjoint (unitary).

If N is a normal operator belonging to a commutative family \mathbf{F} of J -selfadjoint operators, we find a direct sum decomposition of \mathbf{K} which reduces each element of \mathbf{F} and decomposes N into a selfadjoint, J -selfadjoint piece and a “pure J -selfadjoint” piece. Such a normal, pure J -selfadjoint operator S has at least two maximal nonnegative invariant subspaces which reduce every J -selfadjoint operator commuting with S . We determine sufficient conditions for such a subspace to be neutral and the range of a spectral projection for S .

By a “Krein space” we mean a complex Hilbert space \mathbf{K} together with an orthogonal decomposition $\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2$ and the associated bilinear form $(x, y) \rightarrow \langle Jx, y \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{K} and $J = P_1 - P_2$ with P_i representing the orthogonal projection on \mathbf{K}_i . In this setting we may speak of a “selfadjoint operator” (for example) with respect to the inner product or a “ J -selfadjoint” operator H characterized by $\langle JHx, y \rangle = \langle Jx, Hy \rangle$ for all x and y in \mathbf{K} . A “ J -unitary” operator U on \mathbf{K} is an invertible operator satisfying $\langle JUx, Uy \rangle = \langle Jx, y \rangle$ for all x and y in \mathbf{K} . All operators are assumed to be bounded and linear unless specified otherwise, and a space of bounded linear operators is denoted $\mathbf{B}(\)$ in the usual manner. General references for Krein spaces are [2] and [7].

Operators on \mathbf{K} will often be written in the 2×2 matrix form determined by the decomposition $\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2$, which for a J -selfadjoint operator is