

Some Consequences of Rudin's Paper “ L_p -Isometries and Equimeasurability”

WOLFGANG LUSKY

In [9] Rudin investigated the following situation: Let (X, Σ, μ) and $(Y, \tilde{\Sigma}, \nu)$ be positive measure spaces. Assume that E is a subspace of $L_p(\mu)$ and $T : E \rightarrow L_p(\nu)$ is a linear isometry. (All Banach spaces considered here are complex.) Let $e_1, \dots, e_n \in E$ and put $\mathcal{E} = (e_1, \dots, e_n)$, $T\mathcal{E} = (Te_1, \dots, Te_n)$. We shall call \mathcal{E} and $T\mathcal{E}$ *equimeasurable*, if for any Borel set $\Omega \subset \mathbb{C}^n$, $\mu(\mathcal{E}^{-1}(\Omega)) = \nu((T\mathcal{E})^{-1}(\Omega))$. Clearly the equimeasurability of \mathcal{E} and $T\mathcal{E}$ is equivalent to

$$(1) \quad \int_X \psi \circ \mathcal{E} d\mu = \int_Y \psi \circ T\mathcal{E} d\nu$$

for all Borel measurable functions $\psi : \mathbb{C}^n \rightarrow \mathbb{C}$.

One of the main results in [9] is then the following

Theorem (Rudin). *Let μ, ν be finite. Consider $0 < p < \infty$; $p \neq 2, 4, 6, \dots$. Assume furthermore $1_X \in E$, $T 1_X = 1_Y$. Then \mathcal{E} and $T\mathcal{E}$ are equimeasurable.*

Applying this theorem we prove that it is always possible to extend an isometry T from an arbitrary subspace E of L_p ; $p \neq 4, 6, 8, \dots$; to an isometry from a “nice” subspace $F \supset E$. We show that a certain homogeneity property holds for $L_p(0, 1)$; $p \neq 4, 6, 8, \dots$, which is similar to a corresponding property of the Gurarij space, and asserts that all finite-dimensional subspaces of $L_p(0, 1)$, whose geometries coincide, are almost in the same position in $L_p(0, 1)$. It seems surprising that these properties fail to be true in the remaining cases $p = 4, 6, 8, \dots$.

Proposition 1. *Let $1 \leq p < \infty$; $p \neq 4, 6, 8, \dots$; and let the above situation be given (μ, ν arbitrary positive measures). Then there is a subspace $F \subset L_p(\mu)$ with $E \subset F \cong L_p$ and an extension of T to an isometry $\tilde{T} : F \rightarrow L_p(\nu)$.*

Proof. Let $E \subset L_p(\mu)$ and $T : E \rightarrow L_p(\nu)$ be given. Let $e_i \in E$; $i = 1, \dots, 2n$. Since $\text{supp } e_i, \text{supp } Te_i$; $i = 1, \dots, 2n$; are σ -finite we have that

$$\mu\left(\bigcup_{i=1}^{2n} \text{supp } e_i\right) \neq \mu\left(\text{supp}\left(\sum_{i=1}^{2n} \alpha_i e_i\right)\right)$$