

A Convergence Theorem for Solutions of Nonlinear Second Order Elliptic Equations

LAWRENCE C. EVANS

1. Introduction. In this paper we prove a convergence theorem for solutions of general nonlinear, second order, elliptic equations of the form

$$(1.1) \quad \begin{cases} F(u_{x_i x_j}(x), u_{x_i}(x), u(x), x) = f(x) & \text{a.e. } x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth, bounded subset of \mathbb{R}^N ($N \geq 1$). Let us assume that we are given a sequence $u^n \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (for some $p > N$) solving

$$(1.1)_n \quad \begin{cases} F(u_{x_i x_j}^n(x), u_{x_i}^n(x), u^n(x), x) = f^n(x) & \text{a.e. } x \in \Omega \\ u^n(x) = 0 & x \in \partial\Omega, \end{cases}$$

such that the u^n converge *weakly* in $W^{2,p}(\Omega)$ to a limit function u as $f^n \rightarrow f$ uniformly; we will conclude that u solves (1.1).

We note first of all that the weak convergence of a sequence of functions, $\varphi_n \rightharpoonup \varphi$, does not in general imply for a nonlinear ψ that $\psi(\varphi_n) \rightarrow \psi(\varphi)$ in any sense. This difficulty will be circumvented for the situation at hand by an indirect argument analogous to Minty's technique for passing to weak limits (see Minty [7], Browder [4], and especially Chapter 2 in Lions [6]). Minty's device is directly applicable to various quasilinear equations of the form (1.1), for which the formal differential operator $A(u) = F(u_{x_i x_j}, u_{x_i}, u, x)$ can be realized in $L^2(\Omega)$ as a monotone (or pseudomonotone) operator. For the general case however this approach fails, as $A(u)$ will not always then be monotone. Our proof is instead based on the observation that $A(u)$ gives rise to an *accretive* operator on $L^\infty(\Omega)$ (see Appendix 1 for definitions), even when (1.1) is not quasilinear. The main technical difficulty will therefore be to modify Minty's Hilbert space arguments to $L^\infty(\Omega)$.

(We note that certain quasilinear elliptic operators of higher than second order are monotone or pseudomonotone; our arguments depend on the maximum principle and thus do not extend to those cases).

Let us now state our hypotheses on the function $F : \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$: