

Eta Invariants for G-Spaces

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Introduction. Let X^{4k} be a compact oriented Riemannian manifold of dimension $4k$. We suppose that X may have non-empty boundary Y^{4k-1} , of dimension $4k-1$. Moreover we will assume that the metric of X is a product near its boundary Y . By considering the cup product

$$H^{2k}(X, Y) \times H^{2k}(X, Y) \rightarrow H^{4k}(X, Y)$$

and evaluating on the fundamental cycle $[X, Y] \in H_{4k}(X, Y)$ one obtains a symmetric bilinear form on $H^{2k}(X, Y)$. The signature of this bilinear form will be denoted by $\text{Sign}(X)$. Now suppose that $L_k(p_1(\Omega), p_2(\Omega), \dots, p_k(\Omega))$ is the L_k polynomial of Hirzebruch in the Pontriagin forms $p_i(\Omega)$, $1 \leq i \leq k$, of X . The Hirzebruch signature theorem [13] states that the difference

$$(I.1) \quad \int_X L_k(p_1(\Omega), p_2(\Omega), \dots, p_k(\Omega)) - \text{Sign}(X)$$

vanishes if Y is empty. If X is not a closed manifold then the difference (I.1) is not necessarily zero. However it follows from the Hirzebruch signature theorem applied to the double of X that the quantity given by (I.1) depends only on Y and its Riemannian metric.

In their paper [3], Atiyah, Patodi, and Singer established the surprising result that the invariants (I.1) are in fact spectral invariants of the boundary Y . If $\Lambda(Y)$ is the exterior algebra of Y , then let $\Lambda^{\text{ev}}(Y)$ be the subspace of the exterior algebra which consists of forms having type $2p$ for some $0 \leq p \leq 2k-1$. There is a first order self-adjoint elliptic operator $B^{\text{ev}} : \Lambda^{\text{ev}}(Y) \rightarrow \Lambda^{\text{ev}}(Y)$; B^{ev} is naturally defined in terms of the Riemannian geometry of Y . Now B^{ev} has pure point spectrum consisting of eigenvalues λ with multiplicity $\dim(\lambda)$. The spectral function

$$\eta(s, Y) = \sum_{\lambda \neq 0} (\text{sign } \lambda)(\dim \lambda) |\lambda|^{-s}$$

converges for $\text{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s -plane. Moreover, $\eta(0, Y)$ is finite. The central result of [3] is that