

# *Two Notes on Reductions of Ideals*

L. J. RATLIFF, JR. & DAVID E. RUSH

**1. Introduction.** In this paper we give two results on reductions of ideals. (A *reduction* of an ideal  $I$  in a ring  $A$  (commutative with identity) is an ideal  $J \subseteq I$  such that  $JJ^n = I^{n+1}$  for some  $n \geq 1$ .) In Section 2 we show that if  $I$  is a regular ideal in a Noetherian ring  $A$ , then  $I$  is a reduction of

$$I^* = \cup \{I^{i+1} : I^i; i \geq 1\}$$

and, in fact,  $I^{*k} = I^k$  for all large  $k$  and  $I^*$  is the largest ideal in  $A$  with this property. Some properties of the map  $I \rightarrow I^*$  are also given. In the final section we note that if  $f, g \in A[X]$ , then  $c(fg)$  (the content of  $fg$ ) is a reduction of  $c(f)c(g)$ , and give some consequences of this. For example, if  $A$  is a GCD-domain and  $P = (b_0, b_1, \dots, b_n)A$  is a prime ideal that cannot be generated by  $n$  elements, then

$$f = b_0 + b_1X + \dots + b_nX^n$$

generates a prime ideal in  $A[X]$ .

**2.  $I^{*k} = I^k$ , for all large  $k$ .** If  $I$  is an ideal in a Noetherian ring  $A$ , then it follows from the maximal condition that there exist ideals  $I^*$  in  $A$  maximal with respect to  $I^k = I^{*k}$  for all large  $k$ . (2.1) goes considerably farther than this by showing that, in fact, there exists a largest such ideal  $I^*$  and explicitly describing  $I^*$  in terms of  $I$ , when  $I$  is a regular ideal. The ideal  $I$  is therefore a very special kind of reduction of  $I^*$ , since it follows that

$$I^{*k} = II^{*k-1} = \dots = I^{k-1}I^* = I^k.$$

**(2.1) Theorem.** *Let  $I$  be a regular ideal in a Noetherian ring  $A$  and let  $I^* = \cup \{I^{i+1} : I^i; i \geq 1\}$ . Then  $I \subseteq I^*$  and  $I^{*k} = I^k$  for all large  $k$ . Moreover, if  $J$  is an ideal in  $A$  such that  $J^j = I^j$  for all large  $j$ , then  $J \subseteq I^*$ .*

*Proof.* Much as in [Na, (22.1) and (22.2)] there exists a positive integer  $h$  and an element  $x \in I^h$  such that  $x$  is a superficial element of  $I$  of degree  $h$ , and, since  $I$  is regular,  $x$  can be chosen to be a regular element. Thus there exists  $c \geq 1$  such that  $I^k : xA \cap I^c = I^{k-h}$  for all large  $k$ , so it follows from [Na, (3.12)] that  $I^n : xA = I^{n-h}$  for all large  $n$ , say  $n \geq n^*$ . Let  $m = n^* - h$ . Then for all  $i \geq 0$