

Differentiable Maps with Small Critical Set or Critical Set Image

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1.1. Theorem. *Let $f : M^n \rightarrow N^p$ be a C^m ($m \geq n$) proper map, and let $\Sigma(f)$ be its critical set. If $\dim \Sigma(f) \leq n - 2$, and $\dim f(\Sigma(f)) \leq p - 3$, then there is a factorization $f = hg$ such that*

(1) $g : M^n \rightarrow K^p$ is a C^m monotone map (i.e., each $g^{-1}(y)$ is connected) onto the C^m manifold K^p ; and

(2) $h : K^p \rightarrow N^p$ is a k -to-1 diffeo-covering map. Moreover, the factorization is unique, i.e.

(3) if $\bar{h}\bar{g}$ is another such factorization with intermediate space L^p , then there is a C^m diffeomorphism α of K^p onto L^p such that $\bar{g} = \alpha g$ and $\bar{h}\alpha = h$.

In case $n = p$ the hypotheses $m = 3$ and $\dim \Sigma(f) \leq n - 3$ alone imply the conclusion [C-6; (1.1)] (see (5.1)). Examples in §5 show that the dimension hypotheses cannot be improved, and the topological analog is false. The condition C^m may be replaced by C^∞ or real analytic.

Theorem (1.1) is a corollary of a generalization (4.5) of the following result.

1.2. Theorem. *If $f : M^n \rightarrow N^p$ is C^n and proper, and $\dim f(\Sigma(f)) \neq p - 1$, then there is a closed set $Y \subset N^p$ with*

$$\dim Y < \max(0, \dim f(\Sigma(f))),$$

and for every $x \in M^n - f^{-1}(Y)$ there is a neighborhood U of x with $f|U$ topologically equivalent to one of the following proper maps:

(1) $c \times \iota : K^{n-a} \times R^a \rightarrow R^{p-a} \times R^a$, where c is constant, ι is the identity map, and K^{n-a} is closed ($q = 0, 1, \dots, p$).

(2) $(\beta \times \iota)\alpha$, where $\alpha : K^{n-a} \times R^a \rightarrow X \times R^a$ is a monotone map with $\alpha(u, t) = (\alpha_i(u), t)$; X is the one-point union $\vee_i R_i$ of a finite number k of copies of R^{p-a} meeting at 0; for $p - q \neq 2$, $\beta|R_i : R_i \rightarrow R^{p-a}$ is the identity; while for $p - q = 2$, $R^{p-a} = C$ and $(\beta|R_i)(z) = z^{d(i)}$ ($d(i) = 1, 2, \dots$; $q = 0, 1, \dots, p$).

That Y is nicely embedded is shown in (4.9). Analogous results are given for differentiable monotone maps (4.6) and for maps with $n = p$ and non-negative Jacobian determinant (4.7). In case $\dim f(\Sigma(f)) \leq 0$ and $f(\Sigma(f))$ is locally tame, (1.2) is proved in [C-T-1; (1.1)]. In particular, the local tameness hypoth-