

A Generalization of Small Data Techniques and its Application

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Section 1. Introduction. We wish to discuss the small data techniques that were introduced by Segal [4], abstractly formulated by Strauss [5], and clearly expounded by Reed [2] in the analysis of nonlinear wave equations. We are primarily concerned with the application of these methods to the problem of global existence of solutions for given initial data. (The methods can also be applied to scattering theory.) The main point of this paper is that the small data methods have a useful generalization.

The best way to illustrate this generalization is to concentrate on one wave equation. Throughout this discussion we will consider the equation

$$(1.1) \quad \frac{\partial^2 u}{\partial t^2} - \Delta u + u = \lambda u^5$$

in one dimension. For $\lambda < 0$, it is known ([2], [3]) that global solutions exist for many versions of the initial value problem. For $\lambda > 0$, one does not necessarily have global existence of a solution for a given initial pair

$$\left\langle u \Big|_{t=0}, \frac{\partial u}{\partial t} \Big|_{t=0} \right\rangle$$

of functions. Therefore, it is of interest to identify the set of “good” initial data in this case. The basic idea of Segal is that if the initial pairs of functions are sufficiently small in a certain sense, then the corresponding solutions of (1.1) cannot blow up in finite time, in which case global existence of the solutions is guaranteed. His methods are independent of the sign of λ .

We will prove the more general result that if

$$\left\langle v \Big|_{t=0}, \frac{\partial v}{\partial t} \Big|_{t=0} \right\rangle$$

is an initial pair of functions for which there is a global solution v with a nice decay property, then one has global existence of solutions u with the same property for initial data