

Complex Interpolation Spaces, a Discrete Definition and Reiteration

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We shall use definitions and notation in accordance with those of A. P. Calderón's paper [2] on complex interpolation spaces, (see also [1] Chapter 4). In the first part of this note we consider the reiteration formula (paragraph 12.3 [2], 4.6 [1])

$$(1) \quad [[A_0, A_1]_{\theta_0}, [A_0, A_1]_{\theta_1}]_{\sigma} = [A_0, A_1]_s$$

with equality of norms, where θ_0, θ_1 and σ are in $[0, 1]$ and $s = (1 - \sigma)\theta_0 + \sigma\theta_1$. Calderón proved this formula ([2], 32.3) for the case when $A_0 \cap A_1$ is a dense subset of the space $[A_0, A_1]_{\theta_0} \cap [A_0, A_1]_{\theta_1}$. We show here that this density requirement is unnecessary and the result is true for any interpolation pair (A_0, A_1) of Banach spaces.

In the second part of the note we give an equivalent "periodic" or "discrete" definition of the spaces $[A_0, A_1]_{\theta}$.

I. The reiteration formula. To abbreviate the notation let $B_0 = [A_0, A_1]_{\theta_0}$ and $B_1 = [A_0, A_1]_{\theta_1}$. It suffices of course to consider the case $\theta_0 \neq \theta_1$. Let A_j^* denote the dual of A_j , the closed subspace of A_j generated by $A_0 \cap A_1, j = 0, 1$. From [2] 9.3, we have $[A_0, A_1]_j = \bar{A}_j$. The inclusion $[A_0, A_1]_s \subset [B_0, B_1]_{\sigma}$ with

$$\|a\|_{[B_0, B_1]_{\sigma}} \leq \|a\|_{[A_0, A_1]_s}$$

for all $a \in [A_0, A_1]_s$ was established in [2] 32.3. It remains to prove the reverse inclusion and the reverse norm inequality.

We first observe that

$$(2) \quad B_0 \cap B_1 \subset [A_0, A_1]_s.$$

If $s = (1 - \sigma)\theta_0 + \sigma\theta_1$ coincides with θ_0 or θ_1 this is immediate and if $\theta_0 < s < \theta_1$, using elementary properties of real interpolation spaces (4.7 [1], [3] p. 29) we have

$$B_0 \cap B_1 \subset (A_0, A_1)_{\theta_0, \infty} \cap (A_0, A_1)_{\theta_1, \infty} \subset (A_0, A_1)_{s, 1} \subset [A_0, A_1]_s.$$

It is now evident that $B_0 \cap B_1$, which contains $A_0 \cap A_1$, is a dense subspace of both $[A_0, A_1]_s$ and $[B_0, B_1]_{\sigma}$ so to complete the proof of (1) we have only to show