

A Pathology in the Ideal Space of $L(H) \otimes L(H)$

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Introduction. In this paper we establish various results along the lines of those in [15]. If A and B are C^* -algebras $A \odot B$ and $A \otimes B$ will denote their algebraic and spatial C^* -tensor products, respectively. One of our main results, Theorem 4, states that if H is a separable infinite-dimensional Hilbert space and τ denotes the quotient map of $L(H)$ onto the Calkin algebra $A(H) = L(H)/LC(H)$, the homomorphism $\tau \odot \tau : L(H) \odot L(H) \rightarrow A(H) \odot A(H)$ extends to a $*$ -homomorphism of $L(H) \otimes L(H)$ onto $A(H) \otimes_{\eta} A(H)$, where η is a C^* -norm distinct from the spatial C^* -norm. In fact we exhibit a separable C^* -subalgebra D of $L(H)$, containing the compact operators $LC(H)$, such that the restriction of the norm η to $\tau(D) \odot \tau(D)$ differs from the spatial norm. One corollary of this is the surprising result that the ideal $(LC(H) \otimes L(H)) + (L(H) \otimes LC(H))$ of $L(H) \otimes L(H)$ is not maximal. As a further consequence we give an alternative proof of the recent result of J. Anderson [2] that there exists no completely positive lifting of $\tau(D)$ into $L(H)$. In Theorem 12, a strengthening of [15, Corollary 1.8], we furnish an explicit example of a separable C^* -algebra which cannot be embedded as a closed $*$ -subalgebra of a nuclear C^* -algebra. Finally we show how one of our techniques can be used to give a simplified proof of the result of Sakai [10, 4.6.13] that if A is a C^* -algebra containing a closed $*$ -subalgebra B which has a UHF quotient, then A has a type III factor representation.

Our notation and terminology follow [15].

1. The following lemma, which will be used repeatedly, is a generalization of [8, Theorem 3.3] and [15, Proposition 1.3].

Lemma 1. *Let A be a unital C^* -algebra acting non-degenerately on the Hilbert space H , and let π be a non-degenerate representation of a unital C^* -algebra B on H such that $\pi(B) \subseteq A'$. Suppose that the mapping $A \odot B \rightarrow L(H)$; $\sum a_i \otimes b_i \rightarrow \sum a_i \pi(b_i)$ is bounded when $A \odot B$ is endowed with the spatial C^* -norm. Then there is a completely positive unital contraction $\rho : L(H) \rightarrow \pi(B)'$ such that*

$$\rho(axb) = \rho a(x)b \quad (a, b \in A, x \in L(H)).$$