

The Structure of Z_4 Bordism

R. PAUL BEEM

§1. Introduction. The purpose of this paper is to compute some of the structure of the (unoriented) Z_4 bordism ring. Specifically, we will describe some subalgebras of $\mathcal{N}_*(Z_4)$ and get enough structure to determine it as a module over \mathcal{N}_* , the unoriented bordism ring. We also give a few results concerning the algebra structure and find an \mathcal{N}_* basis for the ideal of Z_2 extensions.

In this section we will establish notation, define terms and recall some results which we will need here. In §2, we will compute the kernel of the natural homomorphism from the bordism of “proper” Z_4 actions to unrestricted actions. In §3, we define the subalgebras of $\mathcal{N}_*(Z_4)$ that concern us and prove the main theorem; see Theorem 3.6.

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We will use the notation of [1] and [4]. In particular, $\mathcal{N}_*(H)$ denotes the unoriented bordism of closed, smooth H -manifolds admitting arbitrary isotropy subgroups, $\mathcal{N}_*(H; \mathcal{F}, \mathcal{F}')$ denotes the H -bordism of manifolds whose isotropy subgroups lie in \mathcal{F} and whose boundaries have isotropy subgroups in \mathcal{F}' . $\mathcal{N}_*(H; \mathcal{F}) = \mathcal{N}_*(H; \mathcal{F}, \phi)$ and $\hat{\mathcal{N}}_*(H) = \mathcal{N}_*(H; \{\{1\}\})$ by convention. If $\psi : H \times M \rightarrow M$ is an H action on M , g generates H and $\psi(g, -) = T : M \rightarrow M$, we will denote the bordism class of ψ by $[M, T]$. There are the classes $[S^1, \sqrt{-1}]$ and $[S^{-1}, -1]$ in $\hat{\mathcal{N}}_1(Z_4)$ and $\hat{\mathcal{N}}_1(Z_2)$ which we will denote by y_1 and x_1 respectively.

The family “Proper” is $\{\{1\}, Z_2\}$ as subgroups of Z_4 . There are the extension homomorphisms from the various Z_2 bordism rings to the corresponding Z_4 bordism rings, all of which we will denote by e . e sends $[M, T]$ to $[M \times_{Z_2} Z_4, 1 \times i]$.

We will use ρ to denote the reduction homomorphisms from Z_4 bordism to Z_2 bordism and j_* to denote the natural homomorphism from the various proper Z_4 bordism rings to the unrestricted rings.

Recall from [1] that $\mathcal{N}_*(Z_4; \text{Proper}, \text{Free})$ is the (non-direct) sum of $K_* \oplus y_1 \cdot K_*$ (where K_* is the image of e from $\mathcal{N}_*(Z_2; \text{All}, \text{Free})$ and $y_1 \cdot -$ denotes the twisted product) and $L_* \oplus y_1 \cdot L_*$, where L_* is described as follows:

Let λ_j denote the Hopf (complex, line) bundle over $CP(j)$. Then we have $\lambda_{j_1} \times \cdots \times \lambda_{j_n}$ over $CP(j_1) \times \cdots \times CP(j_n)$. Taking the disk bundle