

# Similarity Cross Sections for Operators

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**1. Introduction.** This note concerns Hilbert space operators  $T$  for which the mapping  $X \rightarrow X^{-1}TX$  admits a locally defined continuous cross section. To make this notion precise we introduce the following terminology. Let  $\mathcal{A}$  denote a Banach algebra with identity and let  $\mathcal{S}(\mathcal{A})$  denote the group of all invertible elements in  $\mathcal{A}$ . For  $T \in \mathcal{A}$ , let  $\mathcal{S}(T)$  denote the similarity orbit of  $T$ , i.e.  $\mathcal{S}(T) = \{X^{-1}TX : X \in \mathcal{S}(\mathcal{A})\}$ . Let  $q : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(T)$  be the norm continuous mapping defined by  $q(X) = X^{-1}TX$ ; a *local cross section* for  $q$  is a pair  $(\varphi, \mathcal{B})$  such that  $\mathcal{B}$  is a relatively open subset of  $\mathcal{S}(T)$  containing  $T$ ,  $\varphi : \mathcal{B} \rightarrow \mathcal{S}(\mathcal{A})$  is norm continuous,  $\varphi(T) = 1$ , and  $q \circ \varphi = 1_{\mathcal{B}}$ , i.e.  $\varphi(S)^{-1}T\varphi(S) = S$  for each  $S$  in  $\mathcal{B}$ . If  $q$  has a local cross section we say that  $T$  has a (local) *similarity cross section*, and in this case  $T$  satisfies the following *sequential similarity lifting property* (hereafter referred to as property (Q)). (Q) If  $\{X_n\} \subset \mathcal{S}(\mathcal{A})$  and  $X_n^{-1}TX_n \rightarrow T$ , then there exists  $\{W_n\} \subset \mathcal{S}(\mathcal{A})$  such that  $W_n \rightarrow 1$  and  $W_n^{-1}TW_n = X_n^{-1}TX_n$  for each  $n$ . In this note we are primarily interested in the case  $\mathcal{A} = \mathcal{L}(\mathcal{H})$ , the algebra of all bounded linear operators on a separable complex Hilbert space  $\mathcal{H}$ .

Our interest in similarity cross sections is motivated by the study of unitary cross sections ([4] [6] [7]), whose principal result we now describe. Let  $\mathcal{U}(\mathcal{H})$  denote the group of all unitary operators on  $\mathcal{H}$ , and for  $T$  in  $\mathcal{L}(\mathcal{H})$  let  $\mathcal{U}(T) = \{U^*TU : U \in \mathcal{U}(\mathcal{H})\}$ , the unitary orbit of  $T$ . Let  $p : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{U}(T)$  be defined by  $p(U) = U^*TU$ . If  $p$  admits a local cross section, then  $T$  is said to have a (local) *unitary cross section*. In [4], D. Deckard and the author obtained the following characterization of operators with unitary cross sections.

**Theorem 1.1.** *For  $T \in \mathcal{L}(\mathcal{H})$  the following are equivalent:*

- i)  $T$  has a local unitary cross section;
- ii)  $T$  satisfies the sequential unitary lifting property [6];
- iii)  $\mathcal{U}(T)$  is norm closed in  $\mathcal{L}(\mathcal{H})$ ;
- iv)  $C^*(T)$ , the  $C^*$ -subalgebra generated by  $T$  and 1, is finite-dimensional;
- v) there exist operators  $A$  and  $B$ , acting on finite-dimensional spaces, such that  $T$  is unitarily equivalent to  $A \oplus B \oplus B \oplus \cdots \oplus B \oplus \cdots$ .

(The equivalence of iii) and iv) is due to D. Voiculescu [13].)

In Theorem 2.2 we prove that each operator satisfying the sequential similarity lifting property is similar to a Jordan operator (defined below). Thus each