

# *The Absence of Killing Fields is Necessary for Linearization Stability of Einstein's Equations*

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**1. Introduction.** The purpose of this paper is to discuss the proofs of Theorems 1 and 2 below.

**Theorem 1.** *Let  $M$  be a compact, connected, oriented smooth manifold,  $\dim M \geq 3$ . Let  $\mathcal{M}$  be the space of  $C^\infty$  [or  $W^{s,p}$ ,  $s > (n/p) + 1$ ] Riemannian metrics on  $M$ ,  $\mathcal{F}$  the  $C^\infty$  [or  $W^{s-2,p}$ ] scalar functions on  $M$  and  $R : \mathcal{M} \rightarrow \mathcal{F}$  the scalar curvature map. Let  $g_0 \in \mathcal{M}$  and  $\rho_0 = R(g_0)$ . Then the equation*

$$R(g) = \rho_0$$

*is linearization stable at  $g_0$  if and only if  $DR(g_0)$  is surjective.*

We recall (Fischer and Marsden [1973]) that a map  $F : X \rightarrow Y$  is called *linearization stable* at  $x_0 \in X$  if for every  $v \in \ker DF(x_0)$ , there is a  $C^1$  curve  $x(t)$  satisfying  $F(x(t)) = y_0 = F(x_0)$ ,  $x(0) = x_0$  and  $x'(0) = v$ .

The 'if' part of Theorem 1 is a consequence of the implicit function theorem. For this part of the theorem there is an important criterion of Bourguignon [1975], Fischer and Marsden [1975a] which states that  $DR(g_0)$  is surjective if  $\rho_0/(n-1)$  is not a constant in the spectrum of  $\Delta_{g_0}$ , the Laplace-Beltrami operator of  $g_0$ .

It is suggested in Fischer and Marsden [1975a] and asserted in Bourguignon, Ebin and Marsden [1976] that the 'only if' part is true as well. The proof outlined is, however, incorrect. We shall point out the error and give the correct proof in §2.

To describe the analogous result in relativity we use the following notation. Let  $V_4$  be a four-manifold and  $M \subset V_4$  an embedded compact three-manifold. Let  $\mathcal{L}$  denote the set of Lorentz metrics on  $V_4$  of class  $C^\infty$  (or  $H^s$ ,  $s > (n/2) + 1$ ) for which  $M$  is a (spacelike) Cauchy surface,  $S_2(V_4)$  the space of  $C^\infty$  (or  $H^{s-2}$ ) symmetric two tensors on  $V_4$  and

$$\text{Ein} : \mathcal{L} \rightarrow S_2$$