

Uniqueness of Solutions to Hyperbolic Conservation Laws

RONALD J. DIPERNA

1. Introduction. We consider strictly hyperbolic systems of conservation laws,

$$(1.1) \quad u_t + f(u)_x = 0, \quad -\infty < x < \infty.$$

Here $u = u(x, t) \in R^n$ and f is a smooth nonlinear mapping from R^n to R^n . The condition of strict hyperbolicity requires that the Jacobian ∇f of f have n real and distinct eigenvalues:

$$\lambda_1(u) < \cdots < \lambda_n(u).$$

By a *weak solution* we shall mean an element of $L^\infty \cap BV(\Omega)$, $\Omega \subset R^2$, which satisfies (1.1) in the sense of distributions. Here $BV(\Omega)$ denotes the space of functions which have locally bounded total variation in the sense of Cesari; *i.e.* the space of $L^1_{\text{loc}}(\Omega)$ functions whose first order partial derivatives are locally finite Borel measures, *cf.* Section 2.

It is well known that the Cauchy problem for (1.1) does not have in general globally defined smooth solutions; the nonlinear structure of the eigenvalues leads to the development of discontinuities in the solution. On the other hand, uniqueness is lost within the broader class of weak solutions; it is possible for many weak solutions to share the same initial data. Thus the problem arises of identifying the class of stable weak solutions.

In this connection, we recall that a function $\eta : \mathcal{D} \rightarrow R$ defined on an open domain $\mathcal{D} \rightarrow R^n$ is an *entropy* for (1.1) with *entropy flux* $q : \mathcal{D} \rightarrow R$ if all smooth solutions with range in \mathcal{D} satisfy an additional conservation law of the form

$$(1.2) \quad \eta(u)_t + q(u)_x = 0.$$

Friedrichs and Lax [13] have observed that most of the conservative systems which result from continuum mechanics (in one and in several space dimensions) are endowed with a globally defined strictly convex entropy. We mention among others the equations of shallow water waves, fluid dynamics, magneto-fluid dynamics, elasticity in certain special cases and the general symmetric hyperbolic system; these systems are recorded in [9] along with their corresponding entropies and constitutive assumptions. In addition Lax [26] has