

# Regularity Properties of Riesz Potentials

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**1. Introduction.** In this paper we shall deal with certain  $L^p$ -estimates of Riesz potentials of  $L^p$ -functions. We shall prove that if  $\mu$  is a measure on  $R^n$  satisfying  $\mu(K) \leq C R_{\alpha,p}(K)$  for all compact sets  $K$  in  $R^n$  then

$$(1) \quad \|I_\alpha f\|_{p,\mu} \leq C \|f\|_p.$$

Here

$$I_\alpha f(x) = \int f(y) |x - y|^{\alpha - n} dy,$$

$R_{\alpha,p}$  is the Riesz capacity (see Section 2 for definition),  $0 < \alpha < n$  and  $1 < p < n/\alpha$ . The result (1) was first proved by Maz'ya ([5] and [6]) for  $\alpha = 1, 2$  and by Adams [1] for the case  $\alpha$  is an integer and  $0 < \alpha < n$ . The point of this paper is the extension to nonintegral values. The proof of (1) is a straightforward consequence of the following theorem, which is our main result.

**Theorem 1.** *If  $\alpha$  is a number such that  $0 < \alpha < n$  and  $1 < p < n/\alpha$  then*

$$\int_0^\infty R_{\alpha,p}(\{x : |I_\alpha f(x)| > \lambda\}) \lambda^{p-1} d\lambda \leq C \|f\|_p^p,$$

where  $C$  is a constant which only depends on  $\alpha, p$  and  $n$ .

We remark that if (1) holds then by considering equilibrium-distributions it is easily seen that  $\mu(K) \leq \text{const } R_{\alpha,p}(K)$ . Also (1) follows from Theorem 1, since

$$\begin{aligned} \|I_\alpha f\|_{p,\mu}^p &= p \int_0^\infty \mu(\{x : |I_\alpha f(x)| > \lambda\}) \lambda^{p-1} d\lambda \\ &\leq C^p \int_0^\infty R_{\alpha,p}(\{x : |I_\alpha f(x)| > \lambda\}) \lambda^{p-1} d\lambda \leq C^p \|f\|_p^p. \end{aligned}$$

We also remark that the restriction  $p < n/\alpha$  is necessary since if  $n/\alpha \leq p \leq \infty$  there is a function  $f \in L^p(R^n)$  such that  $I_\alpha f(x) = \infty$  for all  $x \in R^n$ .

The proof of Theorem 1 is based on the following result.

**Theorem 2.** *Let  $0 < \alpha < n$  and  $1 < p < n/\alpha$ . Let  $\alpha^*$  denote the smallest integer  $\geq \alpha$  and suppose  $\phi : (0, \infty) \rightarrow R$  is  $\alpha^*$  times continuously differentiable*