

On the Boundedness of Solutions of Volterra Equations

GUSTAF GRIPENBERG

1. Introduction and statement of results. The purpose of this paper is to study *a priori* bounds on the solutions of the nonlinear Volterra equation

$$(1.1) \quad x(t) + \int_0^t a(t-s)g(x(s))ds = f(t), \quad t \in R_+ = [0, \infty)$$

where a , g and f are prescribed real functions. In [2] (cf. [6]) this question is considered under the assumptions that $f \in C(R_+; R) \cap BV(R_+; R)$, $g \in C(R; R)$, $a: R_+ \rightarrow R_+$ is nonincreasing and $m\{x | x > 0, g(x) \geq 0\} = m\{x | x < 0, g(x) \leq 0\} = +\infty$. In [7] the condition on a is weakened to $a \in NBV(R_+; R)$, $Re \int_{R_+} e^{-i\omega t} da(t) \geq 0$ for $\omega \in R$, and the crucial condition on the function g is that $|g(x)| \leq k(1 + G(x))$ for some constant k where $G(x) = \int_0^x g(u)du$. Then $\sup_{t \geq 0} |g(x(t))| < \infty$ and if moreover $\lim_{|x| \rightarrow \infty} G(x) = +\infty$ then it also follows that $\sup_{t \geq 0} |x(t)| < \infty$.

In Theorems 1, 2 and 3 below we replace the assumption $f \in BV(R_+; R)$ by $f \in L^\infty(R_+; R)$, but then we must, of course, assume more of the functions a and g . Our first result is

Theorem 1. *Assume that (1.1) holds and that*

$$(1.2) \quad a: R_+ \rightarrow R_+ \text{ is nonincreasing,}$$

$$(1.3) \quad a \in L^1(R_+; R),$$

$$(1.4) \quad g \in C(R; R),$$

$$(1.5) \quad \text{there exist constants } k_1 \text{ and } k_2 \text{ such that}$$

$$|g(x) - g(y)| \leq k_1|x - y| + k_2, \quad x, y \in R,$$

$$(1.6) \quad \text{there exists a constant } k_3 \text{ such that } x > y \text{ implies } g(x) + k_3 \geq g(y)$$

$$(1.7) \quad f \in L^\infty(R_+; R),$$

$$(1.8) \quad x \in L_{loc}(R_+; R).$$