

The Mean Modulus and the Derivative of an Inner Function

PATRICK AHERN

Introduction. In this paper we study the rate at which the average of the modulus of an inner function over a circle of radius r approaches 1 as r approaches 1. Then we give several applications of our results. The paper is divided into six sections. In the first section we get an upper bound on the rate at which the mean modulus of an inner function approaches 1 in terms of the modulus of continuity of the associated singular measure. The result says that the smoother the measure, the slower the growth of the mean modulus. One corollary is a result on exceptional sets. In the second section we show, by exhibiting a class of examples, that the results of the first section are the best possible in terms of smoothness of measure. We also get results from above and below for singular inner functions whose measures are purely atomic, and show by example that these are best possible.

Sections 3–6 are to a large extent applications of the results of the first two sections. In the third section we show how the smoothness of the measure affects the Taylor coefficients of the inner function. In the fourth section we consider the spaces B^p . We see that the spaces B^p to which the derivative of an inner function belongs are determined completely by a condition on the moduli of the Taylor coefficients and that this condition in turn is equivalent to a statement about the distribution of values of the inner function. Using this result we get precise statements about the way that the smoothness of the measure affects the growth of the derivative and the distribution of values of an inner function. Again our examples show that these results are the best possible. In the fifth section we get results on the spaces $A^{q,p}$ to which the derivative of an inner function can belong.

In the last section we turn to the spaces H^p and N . Our main tool here is Theorem 6.1 which says that if φ is an inner function, $0 < p < 1$, and $\varphi' \in H^p$, then $\varphi' \in B^{1/(2-p)}$. It also says that if $\varphi' \in N$, then $\varphi' \in B^{1/2}$. For $0 < p < \frac{1}{2}$, this, used together with the B^p results of Section 5, gives necessary conditions in terms of smoothness of measure, Taylor coefficients, and distribution of values, in order that $\varphi' \in H^p$. As far as we know these are the first necessary