

Mixed-Norm Sobolev Theorems for Lipschitz Spaces

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1. Introduction. Mixed-norm L^p spaces were introduced in [3] by Benedek and Panzone; they have since been studied by a number of authors. In [1], D. R. Adams and the author established mixed-norm Sobolev inequalities for Riesz potentials. Here we investigate similar inequalities for Lipschitz functions.

There are many similarities between [1] and the present work. Indeed, Lipschitz spaces and spaces of potentials are intimately related *via* a number of embedding and restriction theorems; see Taibleson [13]. A detailed comparison of the results presented here with those of [1] helps clarify the relationships between mixed-norm spaces of Lipschitz functions and of Riesz potentials.

We begin by presenting an integral representation for Lipschitz functions. Then we establish Sobolev inequalities by bounding integral operators appropriately. Our representation lends itself naturally to the real method of interpolation. After treating the appropriate extreme cases, we then identify (as far as possible) the relevant intermediate spaces in terms of mixed L^p and Lorentz space norms. Our theorem includes the Sobolev theorem of Herz [7] as a special case and our method gives a particularly simple proof of it.

Finally, we present examples which illustrate the sharpness of our results and which show that the restrictions we have imposed arise naturally and not simply because of a limitation peculiar to our methods. At least in the simpler cases, we show that our results are both as sharp and as general as possible; we have no reason to expect this situation to change in more complicated cases.

2. Preliminaries. For the most part we will work with Lebesgue measurable functions on $R^n = R^{n_1} \times \cdots \times R^{n_m}$, where $n_i > 0$ and $\sum n_i = n$. Elements of R^n will be denoted as $x = (x_1, \cdots, x_m)$ where $x_i \in R^{n_i}$. Euclidean length is denoted $|x|$ and $dx = dx_1 \cdots dx_m$ denotes Lebesgue measure.

L_*^q always denotes the space of Lebesgue measurable functions on $(0, \infty)$ which are in L^q with respect to the measure $t^{-1}dt$. Thus,

$$\|f\|_{L_*^q} = \left\{ \int_0^\infty |f(t)|^q t^{-1} dt \right\}^{1/q}, \quad 1 \leq q < \infty$$