

The Invariance of Partial Isometries

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Introduction. In [1] Marcus proved that if T is a linear transformation on $M_n(\mathbb{C})$, the space of $n \times n$ complex matrices which satisfies

A is unitary implies $T(A)$ is unitary,

then there exists fixed $n \times n$ unitary matrices U, V such that T has the form

$$T(A) = UAV, \quad \text{for all } A \in M_n(\mathbb{C}),$$

or

$$T(A) = UA'V, \quad \text{for all } A \in M_n(\mathbb{C}).$$

In this paper an analogous result is obtained for linear maps T on the $m \times n$ complex matrices ($m < n$) which satisfy

(1) A is a partial isometry implies $T(A)$ is a partial isometry.

The result is contained in the following:

Theorem. *If for some $1 < m < n$*

$$T : M_{m,n}(\mathbb{C}) \rightarrow M_{m,n}(\mathbb{C})$$

is a linear transformation which satisfies (1), then there exists fixed $m \times m$, $n \times n$ unitary matrices U, V such that T has the form

(2) $T(A) = UAV, \quad \text{for all } A \in M_{m,n}(\mathbb{C}).$

Of course, it is clear that the converse of this theorem is also true.

Notation. The space of $m \times n$ complex matrices is denoted by $M_{m,n}(\mathbb{C})$, or $M_n(\mathbb{C})$ when $m = n$. The set of matrices which satisfy

$$AA^* = I_m, \quad \text{the } m \times m \text{ identity}$$

is denoted by $\mu_{m,n}$ or μ_n when $m = n$. If $m \neq n$ these matrices are referred to as *partial isometries*, or *unitary matrices* when $m = n$. For $m < n$, the characterization of partial isometries as being those whose rows are orthonormal (o.n.) will be useful.