

A Note on the Zeros of H^p Functions

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§1. Introduction. Let n be a positive integer larger than 1 and let \mathbb{C}^n denote the complex vector space of all ordered n -tuples $z = (z_1, \dots, z_n)$ of complex numbers, with inner product

$$(1.1) \quad \langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$$

and corresponding norm

$$(1.2) \quad |z| = \langle z, z \rangle^{1/2}.$$

The unit ball B is the set of $z \in \mathbb{C}^n$ with $|z| < 1$, and its boundary is the unit sphere S . There is a unique rotation invariant Borel measure σ on S with $\sigma(S) = 1$.

Following Rudin [4], let ϕ denote a nonconstant, nonnegative, nondecreasing, convex functions defined on $(-\infty, \infty)$. Define $H_\phi(B)$ to be the class of holomorphic functions f on B that satisfy the growth condition

$$(1.3) \quad \sup_{0 < r < 1} \int_S \phi(\log |f(rw)|) d\sigma(w) < \infty.$$

For example, if $\phi(t) = \max(0, t)$, then $H_\phi(B)$ is usually called the Nevanlinna class $N(B)$. If $0 < p < \infty$ and $\phi(t) = \exp(pt)$, then $H_\phi(B)$ is the Hardy class $H^p(B)$. Let $H^\infty(B)$ be the set of all bounded holomorphic functions on B . Then for $0 < p < q < \infty$ we have

$$(1.4) \quad H^\infty(B) \subset H^q(B) \subset H^p(B) \subset N(B).$$

We now give a necessary condition on a zero set to be the zero set of a function in $H_\phi(B)$.

§2. A necessary condition. Let D denote the unit disc in \mathbb{C} . For every $a \in S$ and every holomorphic function f on B define f_a , a holomorphic function on D , by

$$(2.1) \quad f_a(u) = f(ua), \quad \text{for all } u \in D.$$

Assume that $f(0) \neq 0$. Then for $0 < r \leq 1$ define

$$(2.2) \quad n_f(r; a) = \text{the number of zeros, counting multiplicity,}$$

of f_a in $\{u \in D: |u| < r\}$.