

Bounded Mean Oscillation with Orlicz Norms and Duality of Hardy Spaces

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Introduction. The H_p spaces in n dimensions were introduced by Stein and Weiss [13] in analogy with the classical theory for Hardy spaces. The problem of finding the dual space of H_p , $1 < p < \infty$, is trivial, since H_p is identical with L_p for $1 < p < \infty$. The dual space of H_1 is BMO . This was discovered by C. Fefferman [4].

The aim of this paper is to generalize Fefferman's result to Hardy spaces H_φ with more general Orlicz norms. As the main result we get that the dual space of H_φ is $L_{\varphi^*}^\#$, where φ and φ^* are convex conjugate functions and $L_{\varphi^*}^\#$ is a generalization of $BMO \equiv L_{\varphi^*}^\#$. This is under the assumption that φ does not increase too fast (Theorem 5.1).

We also consider different ways of describing the oscillation of a function f , by using estimates of the following kind:

$|\{x \in Q; |f(x) - c_Q| > A\}| < s|Q|$ for $A > 0$ and cubes Q in \mathbb{R}^n and with s fixed, $0 < s < 1$.

This gives rise to alternative definitions of BMO with very weak conditions, which, in fact, appeared in an old but not very well-known paper by John ([4]).

In Section 1 we define the Orlicz spaces (L_φ) and list some of their properties, which we shall use later.

In Section 2 we show some interpolation theorems for Orlicz spaces.

The $L_{\varphi^*}^\#$ -spaces are defined in Section 3, as a generalization of BMO , by means of some maximal operators. We consider the equivalence of different definitions. We also give the necessary and sufficient conditions for φ^* , in order that $L_{\varphi^*}^\#$ is strictly larger than L_{φ^*} .

In Section 4 we give the definition of the Riesz transforms and show some of their properties. We also define the Hardy spaces with Orlicz norms and show their connections with the Riesz transforms.

The main result is proved in Section 5. We also get some different characterizations of $L_{\varphi^*}^\#$ (Theorem 5.2) and a necessary and sufficient condition for a harmonic function on \mathbb{R}_+^{n+1} to be representable as the Poisson integral of a function in $L_{\varphi^*}^\#$ (Theorem 5.3). The latter is a generalization of a result of Fabes, Johnson and Neri [3]. Theorem 5.4 is an estimate of an integral of a