

Multiple Positive Fixed Points of Nonlinear Operators on Ordered Banach Spaces

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1. Introduction. In this paper we consider the existence of multiple positive fixed points of completely continuous nonlinear operators defined on the cone K of an ordered Banach space E . Our main results give sufficient conditions for such an operator to have two, and in some cases three, positive fixed points.

The methods developed here improve a well-known multiple fixed point technique formulated by Krasnosel'skii and Stecenko [6] for certain boundary value problems and Hammerstein integral operators. Those authors considered an operator A bounded above and below by suitable order-preserving operators A_1 and A_2 . They showed that if A_1 and A_2 have alternating sections of rapid and slow growth, it is sometimes possible to find disjoint order intervals (that is, sets of the form $\langle x, y \rangle = \{z \in K : x \leq z \leq y\}$) left invariant by A . If order intervals in K are bounded, the existence of a fixed point of A in each of the A -invariant order intervals then follows from Schauder's fixed point theorem. Using the fixed point index, Amann [1] has shown, under assumptions very similar to those in [6], that additional "intermediate" fixed points can sometimes be found outside the invariant order intervals. Results closely related to those of [6] and [1] can be found in [2], [3], [7], [8], and [9]. (Also, see [10].)

Here we investigate the existence of multiple positive fixed points of operators that need not satisfy the stringent monotonicity and growth assumptions imposed by the methods of [6]. In fact, the results proved here require no monotonicity assumptions whatever on the operator A . Hence they allow the determination of multiple fixed points that stem, in some sense, from a "change in direction" of A in the cone as well as in a change in the growth rate of A .

Consider, as an example, the nonlinear boundary value problem

$$(1.1) \quad x''(t) = -f(x(t)), \quad 0 \leq t \leq 1,$$

$$(1.2) \quad x(0) = 0 = x'(1),$$

where f maps \mathbf{R} continuously into \mathbf{R} . The solutions of (1.1)–(1.2) are the fixed points of the operator A defined on $C[0, 1]$, the space of real-valued, continuous functions on $[0, 1]$, by