

Hyponormal Operators and Spectral Multiplicity

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1. Introduction. Let T be a bounded hyponormal operator on the separable Hilbert space \mathfrak{H} , thus

$$(1.1) \quad T^*T - TT^* = D \geq 0,$$

or, equivalently, if T has the Cartesian representation $T = H + iJ$,

$$(1.2) \quad HJ - JH = -iC, \quad \text{where } C = (1/2)D \geq 0.$$

It is known that $\sigma(H)$ and $\sigma(J)$ are precisely the projections of $\sigma(T)$ onto the real and imaginary axes; see [9], p. 46. If H has the spectral resolution $H = \int t dE_t$ and if for any bounded operator A on \mathfrak{H} one defines $A_\Delta = E(\Delta)AE(\Delta)$ on $E(\Delta)\mathfrak{H}$, where Δ is any open interval of the real line, then it follows from (1.2) that $H_\Delta J_\Delta - J_\Delta H_\Delta = -iC_\Delta$. Consequently, T_Δ is hyponormal on $E(\Delta)\mathfrak{H}$ and its spectrum lies in the closure of the strip $\{z: \operatorname{Re}(z) \in \Delta\}$.

It was shown in Putnam [10] that

$$(1.3) \quad \pi \|D\| \leq \operatorname{meas}_2 \sigma(T)$$

and that

$$(1.4) \quad \sigma(T_\Delta) \subset \sigma(T).$$

(In case D is compact these relations were established by Clancey [2].) In Putnam [11], it was shown that (1.4) can be refined to

$$(1.5) \quad \sigma(T_\Delta) \cap \{z: \operatorname{Re}(z) \in \Delta\} = \sigma(T) \cap \{z: \operatorname{Re}(z) \in \Delta\}.$$

Further, as a consequence of (1.5), or even of (1.4), if B is any Borel set of the real line and if $F(t)$ is the one-dimensional measure of the vertical cross section $\sigma(T) \cap \{z: \operatorname{Re}(z) = t\}$ of $\sigma(T)$, then

$$(1.6) \quad \pi \|E(B)DE(B)\| \leq \int_B F(t) dt.$$

A hyponormal operator T is said to be completely hyponormal if there is no non-trivial reducing space of T on which T is normal. In this case, the operator