

Base Spaces

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0. Introduction. If one writes out all the requirements needed for the homotopy classification of complex vector bundles *and* for the direct construction of their Chern classes, formulated in terms of the base spaces of the bundles, one reaches a natural category \mathcal{B} of those spaces which serve as base spaces. The category \mathcal{B} contains the traditional category \mathcal{W} of spaces of the homotopy types of CW spaces.

One of the principal features of the category \mathcal{B} is that it is a natural category in which theorems can be proved by the classical Mayer-Vietoris technique. For example, one uses the Mayer-Vietoris technique to prove the Leray-Hirsch theorem for bundles over spaces in \mathcal{B} ; the corresponding result for \mathcal{W} then follows from the inclusion $\mathcal{W} \subset \mathcal{B}$.

In §1 we define \mathcal{B} and establish its principal features, including the “Mayer-Vietoris technique”. In §2 we verify the inclusion $\mathcal{W} \subset \mathcal{B}$. In §3 we show that \mathcal{B} (and \mathcal{W}) have all the properties needed for the homotopy classification of complex vector bundles and for the direct construction of Chern classes. The remarks of §4 are self-explanatory.

1. The category \mathcal{B} . The category \mathcal{B} will be defined in two stages, the first being an induction on the natural numbers $n \geq 0$. In describing the inductive step we shall suppose in part that a given topological space X can be covered by q families of open sets, for some natural number $q > 0$, where the sets in each one of the q families will be mutually disjoint; that is, for q unrelated index sets B_1, \dots, B_q there will be q families $\{U_{\beta_1}; \beta_1 \in B_1\}, \dots, \{U_{\beta_q}; \beta_q \in B_q\}$ of open sets $U_{\beta_p} \subset X$ which collectively cover X , and for each fixed $p \leq q$ the sets in the family $\{U_{\beta_p}; \beta_p \in B_p\}$ will be disjoint from one another. For notational convenience we let $\{U_{1,\alpha}\}_\alpha, \dots, \{U_{q,\gamma}\}_\gamma$ represent the q families of open sets, without explicitly identifying the index sets; for each $p \leq q$ the notation $\{U_{p,\beta}\}_\beta$ will represent the p^{th} family.

1.1 Definition. A topological space is of 0^{th} type if it is a disjoint union of contractible spaces. Suppose that a topological space X can be covered by the union of finitely many families $\{U_{1,\alpha}\}_\alpha, \dots, \{U_{q,\gamma}\}_\gamma$ of open sets $U_{p,\beta} \subset X$ such that for each $p \leq q$ the sets $U_{p,\beta}$ in the family $\{U_{p,\beta}\}_\beta$ are disjoint from one another. Then X is of n^{th} type whenever there is such a covering with the addi-