

# On the Structure of Certain $H^2(\mu)$ Spaces

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Let  $\mu$  be a finite Borel measure with compact support in the complex plane, and denote by  $H^2(\mu)$  the closure in  $L^2(\mu)$  of the complex polynomials. For seemingly simple choices of  $\mu$ ,  $H^2(\mu)$  can be quite mysterious. In this paper we investigate the structure of  $H^2(\mu)$  for a particular class of measures  $\mu$  supported on the closed unit disk  $\bar{D} = \{z: |z| \leq 1\}$ .

Let  $\nu$  denote the part of  $\mu$  carried by the open disk  $D$  and suppose  $w \, d\sigma + \mu_s$  is the Lebesgue decomposition of the part of  $\mu$  carried by the unit circle  $\partial D$ ; here  $d\sigma = (2\pi)^{-1}d\theta$  is normalized Lebesgue measure,  $0 \leq w$  is in  $L^1(\sigma)$  and  $\mu_s \perp \sigma$ . Since  $L^2(\mu_s)$  is always a direct summand of  $H^2(\mu)$  [4, p. 28], we may assume that  $\mu_s = 0$  and  $\mu = \nu + w \, d\sigma$ . We study conditions under which

$$(1) \quad H^2(\mu) = H^2(\nu) \oplus L^2(w \, d\sigma),$$

in which case we say that  $H^2(\mu)$  splits. This will happen if and only if the characteristic function of  $\partial D$  lies in  $H^2(\mu)$ . Of course, the precise meaning of (1) is that the natural isometry

$$f \rightarrow \chi_D f \oplus \chi_{\partial D} f$$

of  $H^2(\mu)$  into  $H^2(\nu) \oplus L^2(w \, d\sigma)$  is actually onto,  $\chi_E$  being the characteristic function of a set  $E$ .

We consider mainly the case in which  $\nu$  is circularly symmetric:  $d\nu = dm(r)d\theta$ , or more precisely,

$$\int f \, d\nu = \int_0^{2\pi} \int f(re^{i\theta}) dm(r) \, d\theta$$

for some Borel measure  $m$  on  $[0,1)$ . Then  $H^2(\nu)$  has a simple structure as a space of analytic functions and (1) presents rather clearcut difficulties. This problem seems to have been first considered by E. M. Kegejan [9, 10] who obtained some conditions necessary and some sufficient for splitting when  $d\nu = r \, dr \, d\theta$  (area measure on  $D$ ) and  $w = \chi_E$  with  $E$  a subset of  $\partial D$ . S. V. Hruščëv gave a complete solution [8] in this setting with  $w = \chi_E$ ,  $d\nu = dm(r) \, d\theta$  and  $m$  "slowly decaying" near  $r = 1$ . Meanwhile, W. Hastings [6] constructed an example showing that  $H^2(\mu)$  can split when  $w$  is strictly positive and  $dm(r) = G(r)r \, dr$ ,  $G(r) > 0$  on  $[0,1)$ . Subsequently, A. L. Vol'berg showed that