

Commutants of Triangular Matrix Operators

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1. Introduction. It is well known that if S is a weighted shift operator on a Hilbert space, then

(1) the commutant of S is algebraically isomorphic to a space of formal power series and

(2) the commutant of S is the strong closure of the set of polynomials in S .

A good treatment of these results is given in the paper of Shields and Wallen [9]. Banach space versions have been given by several authors [1], [4], [5], [6]. In a recent paper, Crownover and Hansen [2] studied the commutants of generalized integration operators on the space of analytic functions on the open unit disc in the plane. They showed that (1) holds also for this class of operators.

A property held in common by all the operators mentioned above is the possession of a lower triangular matrix representation with respect to some basis. It is of interest to ask whether the algebraic result (1) can be obtained for a general class of operators which are triangular in some sense and which include the shift operators previously studied.

It is instructive to consider the finite dimensional case. Suppose S is a strictly lower (upper) triangular ($n \times n$) matrix with the property that every entry on the main subdiagonal (superdiagonal) is nonzero. We give an elementary proof of the fact that every $n \times n$ matrix T which commutes with S is necessarily a polynomial in S .

With this finite dimensional version as a model, we can define classes of upper matrix operators and lower matrix operators in such a way that (1) holds under fairly general conditions.

The conclusion (2) is much more difficult. We can obtain it for certain restricted classes of operators on special spaces. In some instances we can identify the commutant as either isomorphic to the underlying space or to its dual space.

2. The finite dimensional case.

(2.1) Theorem. Let $B = (b_{ij})$ be a strictly upper triangular $n \times n$ matrix with $b_{j,j+1} \neq 0$ for $j = 1, 2, \dots, n-1$. If T is an $n \times n$ matrix such that $TB = BT$, then there is a polynomial $p(z) = \sum_{k=0}^{n-1} c_k z^k$ of degree less than or equal to $n-1$ such that $T = p(B)$.