

Operator Ranges and Invariant Subspaces

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1. Introduction. Suppose that \mathcal{A} is an algebra of bounded operators on a Banach space X , and that C and D are bounded linear operators on X . We do not assume that \mathcal{A} is uniformly closed, but, to avoid trivial special cases, we always implicitly assume that \mathcal{A} contains an operator which is not a multiple of the identity. In a series of recent papers [14] [16] [6], Fong, Nordgren, Radjabalipour, Radjavi, and Rosenthal give a variety of conditions on \mathcal{A} , C , and D under which the assumption that $\mathcal{A}C \subseteq D\mathcal{A}$ implies that \mathcal{A} or its commutant, \mathcal{A}' , has an invariant subspace. Their most definitive results are in [6], which also contains applications proving that certain operators have hyperinvariant subspaces. In the present paper, we will extend the results of Fong *et al* [6] and apply their results and our extensions to prove the existence of hyperinvariant subspaces of a wide class of quasi-nilpotent operators.

To describe our results about operator algebras, we will need some preliminary definitions. We say that \mathcal{A} is an *operator range algebra* if it is the range of a bounded linear operator from some Banach space to the algebra of bounded operators on X (cf. [6, Section 3]). Thus \mathcal{A} is an operator range algebra precisely when it is the set of left multiplications for some Banach algebra over which X is a Banach module [11, page 413]. A sequence of non-zero projections $\{P_n\}$ will be called a *decomposition at 0* of D provided that each P_n commutes with D and that the spectral radius of $P_n D$ approaches 0. If D is noninvertible it has a decomposition at 0 if its spectrum is countable, or, more generally, if $\{0\}$ is a component of the spectrum of D [13, Corollary 1, page 83], or if D is a spectral operator [5, Definition 5, page 1931].

In [6, Theorems 3 and 4] Fong *et al* prove that if \mathcal{A} is an operator range algebra, if C is compact and D has a decomposition at 0, and if $\mathcal{A}C \subseteq D\mathcal{A}$, then \mathcal{A} has an invariant subspace; and, if D is also compact, then \mathcal{A}' has an invariant subspace. In Theorem (3.1), we obtain the same results when $C\mathcal{A} \subseteq \mathcal{A}D$ (when X is reflexive and D is compact, our Theorem (3.1) reduces to [6, Corollary 3]). In Theorem (3.2), we obtain similar results about joint invariant subspaces for finite collections of operator range algebras. The results about joint invariant subspaces are based on Corollary (2.3), below, which is a generalization to multilinear operators of the results in [6, Theorem 2] on ranges of linear operators.