

# Possible Structures for the Set of Cyclic Vectors

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Let  $T$  be a cyclic operator acting on a complex separable infinite dimensional Banach space  $\mathcal{X}$ , and let  $\mathcal{C}(T)$  and  $\mathcal{N}(T)$  denote the set of all cyclic vectors of  $T$  and its complement in  $\mathcal{X}$ , respectively.  $\mathcal{C}(T)$  has nonempty interior if and only if it is open; if  $\mathcal{C}(T)$  is closed in  $\mathcal{X} \setminus \{0\}$ , then either  $\mathcal{C}(T) = \mathcal{X} \setminus \{0\}$  or  $\mathcal{C}(T)$  is nowhere dense. Several conditions are found to be equivalent to 1)  $\mathcal{N}(T)$  has nonempty interior; among them 2) The point spectrum of  $T^*$  has nonempty interior, 3)  $T^*$  does not have the single-valued extension property, 4) There exists a bounded linear mapping  $X: \mathcal{X} \rightarrow \mathcal{A}(\Omega) = \{f: f \text{ is continuous on } \Omega^- \text{ and analytic on } \Omega\}$  (sup norm topology) with a dense range, where  $\Omega$  is a Jordan domain contained in the spectrum of  $T$ , such that  $XT = M_z X$  ( $M_z =$  multiplication by  $z$ ). Similar results hold for analytically cyclic operators.

**1. Introduction.** Let  $\mathcal{L}(\mathcal{X})$  denote the algebra of all (bounded linear) operators acting on the complex Banach space  $\mathcal{X}$ . Given  $T \in \mathcal{L}(\mathcal{X})$ , we shall denote by  $\mathcal{A}(T)$ ,  $\mathcal{A}^a(T)$  and  $\mathcal{A}'(T)$  the weak closure of the polynomials in  $T$ , the weak closure of the rational functions in  $T$  poles outside the spectrum  $\sigma(T)$  of  $T$  (i.e., the ‘‘analytic algebra’’ of  $T$  [17]) and the commutant of  $T$ , respectively.

Given a (weakly closed, identity containing) subalgebra  $\mathcal{A}$  of  $\mathcal{L}(\mathcal{X})$ , the *multiplicity* (*strict multiplicity*, respectively) of  $\mathcal{A}$  is defined by

$$\mu(\mathcal{A}) = \inf\{\mathfrak{t}(\Gamma): \mathcal{X} = \bigvee [Ax: A \in \mathcal{A}, x \in \Gamma]\}$$

$$(\bar{\mu})(\mathcal{A}) = \inf\{\mathfrak{t}(\Gamma): \mathcal{X} = \text{linear span}[Ax: A \in \mathcal{A}, x \in \Gamma]\},$$

respectively), where  $\bigvee$  denotes ‘‘the closed linear span of’’.

Throughout Sections 2 and 3,  $T$  will always denote a *cyclic operator*, i.e., an operator  $T \in \mathcal{L}(\mathcal{X})$  such that  $\mathcal{A}(T)$  has multiplicity 1; this condition is clearly equivalent to  $\mathcal{X} = \bigvee_{k \geq 0} T^k y$ , or  $\mathcal{X} = [\mathcal{A}(T)y]^-$  (where the upper bar denotes *norm-closure*), for some vector  $y$ . In this case,  $y$  is called a *cyclic vector* for  $T$  (or, for  $\mathcal{A}(T)$ ). If  $\mathcal{X}$  is actually equal to  $\mathcal{A}(T)y$ , then  $T$  is called a *strictly cyclic operator* and  $y$  is a *strictly cyclic vector* for  $T$  [20]. Similarly,  $A \in \mathcal{L}(\mathcal{X})$  is called *analytically cyclic* (*analytically strictly cyclic*, respectively) if  $\mathcal{X} = [\mathcal{A}^a(T)y]^-$  ( $\mathcal{X}$