

Complete Integrability and Perturbation of a Nonlinear Dirichlet Problem, I

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In this article we study a class of nonlinear elliptic operators \mathcal{A} defined on a bounded domain $\Omega \subset \mathbb{R}^N$, supplemented by null Dirichlet boundary conditions. Regarding each \mathcal{A} as a mapping of an appropriate separable infinite dimensional Hilbert space H into itself we show that each mapping \mathcal{A} is globally equivalent (in a sense to be made precise later) to the mapping $\mathcal{A}_0: (t, \omega) \rightarrow (t^2, \omega)$. Here (t, ω) denotes a canonical orthogonal decomposition of H into a fixed one dimensional subspace S_1 whose Cartesian coordinate is denoted by t and its orthogonal complement whose coordinate is denoted by ω . This fact implies the integrability of the nonlinear problem

$$(1) \quad \mathcal{A}(u) = g \quad g \in L_2(\Omega)$$

(supplemented by null Dirichlet boundary conditions), in the sense that formulae for the explicit solutions of (1) can be written down in terms of canonical global coordinate changes. Since we show \mathcal{A} is equivalent to \mathcal{A}_0 , the mapping \mathcal{A} is certainly not a global homomorphism. Thus the structure of the solutions of (1) varies as g changes in $L_2(\Omega)$, and this fact is reflected in our explicit formula.

Our methods are sufficiently general to study the effect of perturbations of the boundary value problem (1). Indeed for sufficiently small “smooth” perturbations of \mathcal{A} , an interesting stability property prevails, namely the perturbed mapping $\tilde{\mathcal{A}}$ is still globally equivalent to \mathcal{A}_0 . However for less smooth small perturbations, we show (roughly speaking) that outside a small neighbourhood of the singular values the number of solutions of the perturbed problem is identical with those of (1). Moreover, in this case the solutions of (1) are good approximations to those of the perturbed problem.

The class of nonlinear elliptic boundary value problems we consider was originally studied by Ambrosetti and Prodi in [1]. By general arguments they showed relative to (1) that $\mathcal{A}^{-1}(g)$ has exactly zero or two points if g is not a singular value for \mathcal{A} and exactly one point otherwise. In [2] Berger and Podolak gave an explicit Cartesian representation of the singular values of \mathcal{A} and used this fact to derive explicit criteria to determine the number of solutions of (1) for a given g . See Nirenberg [3, pages 111–126] for a review of these papers. The