

# Rank 2 Bundles over $P^n$ and the $e$ -Invariant

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**1. Introduction and statement of results.** In the course of investigating rank 2 complex vector bundles over the  $n$ -dimensional complex projective space  $\mathbb{C}P^n$  (cf. [4], [5]) we encounter the following problem. Let  $\eta: \mathbb{C}P^n \rightarrow BU(2)$  be the classifying map of a rank 2 bundle and  $h: S^{2n+1} \rightarrow \mathbb{C}P^n$  the Hopf map. The adjoint of

$$S^{2n+1} \xrightarrow{h} \mathbb{C}P^n \xrightarrow{\eta} BU(2)$$

is a map  $S^{2n} \rightarrow U(2)$  which factors through  $SU(2) \cong S^3$ ; what is the complex  $e$ -invariant of the resulting element in  $\pi_{2n}S^3$ ? Can one express it in terms of the Chern classes  $c_1, c_2$  of  $\eta$ ?

In solving this problem we were helped significantly by the information from Larry Smith that George Cooke had solved the corresponding problem for symplectic line bundles over the quaternionic projective space  $\mathbb{H}P^n$ . Smith could not find a proof of Cooke's result among his papers, but the author was able to construct one which can be carried over to the complex case. Cooke's theorem appears here as Theorem 4.

As  $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$  with  $x \in H^2(\mathbb{C}P^n, \mathbb{Z})$ , the Chern classes  $c_1\eta, c_2\eta$  of a rank 2 bundle  $\eta \downarrow \mathbb{C}P^n$  are integer multiples of  $x$  respectively  $x^2$ . Therefore we shall regard them as integers. We write

$$c_1 = \delta_1 + \delta_2 \quad c_2 = \delta_1 \cdot \delta_2$$

for  $\delta_1, \delta_2$  complex numbers (the roots of the polynomial  $y^2 - c_1y + c_2$ ). In [2] Schwarzenberger demonstrates that the Chern classes  $c_1 = c_1\eta, c_2 = c_2\eta$  of a rank 2 bundle  $\eta \downarrow \mathbb{C}P^n$  must satisfy

$$S_n: \binom{\delta_1}{k} + \binom{\delta_2}{k} \in \mathbb{Z} \quad 2 \leq k \leq n.$$

$\left( \binom{\delta_1}{k} + \binom{\delta_2}{k} \right)$  is symmetric in  $\delta_1, \delta_2$ ; thus this expression represents a polynomial in  $c_1$  and  $c_2$  with rational coefficients. ) Because the methods are similar to those needed for the  $e$ -invariant calculations we give a proof of the following theorem of Alan Thomas [6].