

A Localization Theorem for Toeplitz Determinants

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§1. Introduction. This paper is concerned with the asymptotic behavior of Toeplitz determinants, that is $D_n[\phi] = \det T_n[\phi]$ where $T_n[\phi] = (\phi_{j-k})$, $0 \leq j, k \leq n$, $\phi \in L^1(S')$ and $\phi_{j-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\theta) e^{-i(j-k)\theta} d\theta$. More specifically it will be concerned with the following question: Suppose the asymptotic behaviors of $D_n[\phi]$ and $D_n[\psi]$ are known. What can be said about the asymptotic behavior of $D_n[\phi\psi]$? An answer will be given in the case where the Toeplitz operators $T_f: H^2 \rightarrow H^2$, $f \in L^\infty$ defined by $T_f(g) = P(fg)$ satisfy certain properties. Let \mathcal{J} denote the set of functions $f \in L^\infty$ whose Toeplitz matrices $T_n[f]$ are invertible for sufficiently large n and which have the property that $T_n[f]^{-1} \rightarrow T_f^{-1}$ strongly. The implication in the last statement is that T_f^{-1} must also exist. For functions $\phi, \psi \in \mathcal{J}$ also satisfying the condition that the operators $T_\psi^{-1} T_{\phi\psi} T_\phi^{-1}$ and $T_\phi^{-1} T_{\phi\psi} T_\psi^{-1}$ are determinant class we will show (Theorem 2.1) that

$$\lim_{n \rightarrow \infty} \frac{D_n[\phi\psi]}{D_n[\phi]D_n[\psi]} = \det(T_\phi^{-1} T_{\phi\psi} T_\psi^{-1})(T_\psi^{-1} T_{\phi\psi} T_\phi^{-1}).$$

We should remark here that the condition that $T_\psi^{-1} T_{\phi\psi} T_\phi^{-1}$ is determinant class is clearly equivalent to the operator $T_{\phi\psi} - T_\psi T_\phi$ being trace class. This will be true if the functions ϕ and ψ are not both “bad” at the same time. (See [Wf]). Thus the main theorem of the paper can be thought of as a localization result for Toeplitz determinants.

An application of this result will be the asymptotic formula for Toeplitz determinants with piece-wise continuous symbol. This will be done in §3 and will require the piece-wise continuous function to have index zero and be sufficiently smooth at points of continuity. The use of Theorem 2.1 simplifies much of the previous work done in this direction (see [W2], [B]).

This theorem is also related to work done in [B-H]. A similar type of quotient theorem is proved there and it has applications to Toeplitz matrices with unbounded generating functions. The interested reader is referred to that paper for more information. However, the results of §2 lend themselves better to the piece-wise continuous case and the proof of Theorem 2.1 is very direct.