

*On The Discrete Part of a Measure
Preserving Transformation and a Paper by
Quiring and Turek*

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Quiring and Turek [1] constructed an ergodic measure preserving transformation τ on $[0, 1]$ with the following properties: 1. There exists a sequence of integers h_n such that for all n $\lambda_n = \exp(2\pi i/h_n)$ is an eigenvalue of τ , *i.e.* there exist measurable functions f_n into the unit circle of the complex plane such that $f_n \circ \tau = \lambda_n f_n$. 2. There exists an irrational number α such that $\lambda = \exp(2\pi i\alpha)$ is an eigenvalue of τ . We denote its eigenfunction by f . They then assert that τ does not have purely discrete spectrum, *i.e.* the eigenfunctions do not form a basis of L_2 . They prove this by asserting that there exists an n such that f_n and f are not independent in the probabilistic sense. In this note we prove the following which contradicts this last assertion.

Theorem. *Let f and g be eigenfunctions of an ergodic measure preserving transformation τ on a probability space with eigenvalues λ and μ respectively. If $\{\lambda^n: n \in \mathbb{Z}\} \cap \{\mu^n: n \in \mathbb{Z}\} = \{1\}$, then f and g are independent in the probabilistic sense.*

The error in [1] is the statement on page 497 that "An eigenfunction g for $\exp(2\pi i/h_k) \cdots$ has constant value (say 1) on a set \cdots which includes $C_k(1)$." This statement is only true with $C_k(1)$ replaced by $C_m(1)$ for $m > k$.

Lemma. *Let τ be an ergodic measure preserving transformation on a probability space (X, \mathcal{F}, P) . Let Λ be a subgroup of the group of eigenvalues of τ , G be the group of eigenfunctions corresponding to Λ , and $\mathcal{F}(\Lambda)$ the smallest σ -algebra such that f is $\mathcal{F}(\Lambda)$ -measurable for all $f \in G$. Then τ acting on $(X, \mathcal{F}(\Lambda), P)$ has discrete spectrum Λ .*

Proof. Let \mathcal{A} denote the set of all finite linear combinations (with complex coefficients) of functions in G . Because G is a group of bounded functions closed under complex conjugation, \mathcal{A} is an algebra of bounded functions closed under complex conjugation. Let f_1, \cdots, f_n be real valued functions in \mathcal{A} . Since $f = (f_1, \cdots, f_n)$ is bounded, its range is contained in a finite rectangle $[-a, a]^n$. For a Borel set $E \subseteq [-a, a]^n$ define $\mu(E) = P(f^{-1}(E))$. Then μ is a finite Borel