

# Smooth Circle Actions on Manifolds

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1. Let  $G$  denote the circle group  $S^1$ . In [4], Edmonds and Fintushel determined which p.l. maps  $f: X \rightarrow Y$  give rise to an orbit map  $F: X \rightarrow X/G = Y$ . In this paper, we consider a smooth  $G$ -action on a compact orientable manifold  $X^{n+1}$  without fixed point. It follows from the Slice theorem that the orbit space  $H = X/G$  is a space locally homeomorphic to the orbit space of an orthogonal action on the Euclidean space  $R^n$  by a finite cyclic group.

The purpose of this note is to show that any such space, which is called a  $V$ -manifold (see Section 2 below), can be realized as the orbit space of a  $G$ -action on some smooth manifold  $M^{n+1}$ .

For simplicity, we assume that all the manifolds considered here are without boundaries. The author wishes to thank Ted Chang for useful comments and the referee for many helpful suggestions.

2. We recall here some definitions concerning  $C^\infty$   $V$ -manifolds, and refer the readers to [1], [8] and [10] for details.

Let  $G_U$  be a finite group acting linearly on  $R^n$  with the fixed point set of dimension at most  $n - 2$ ,  $U^*$  an invariant open subset of  $R^n$ , and  $q_U: U^* \rightarrow U = U^*/G_U$  the projection map satisfying  $q_U \circ g = q_U$  for all  $g \in G_U$ . Then we call  $U$  a local  $V$ -manifold. A  $V$ -map  $f$  from one local  $V$ -manifold  $U$  to another  $W$  is a  $(G_U, G_W)$ -equivariant map  $f^*$  from  $U^*$  to  $W^*$ ; that is, there exists a homomorphism  $\lambda: G_U \rightarrow G_W$  such that  $\lambda(g) \circ f^* = f^* \circ g$  for  $g \in G_U$ .

A  $V$ -manifold structure on a Hausdorff paracompact space  $X$  consists of an open covering of  $X$  by local  $V$ -manifolds  $\{U\}$  such that the overlap maps are  $V$ -diffeomorphisms. Each  $\{U^*, G_U\}$  (or  $\{U^*, G_U, q_U\}$ ) is called a local uniformizing system (l.u.s.) for  $U$  and  $G_U$  a local group [10]. A  $C^\infty$ -map is given by a compatible family of equivariant maps on l.u.s.'s.

We also have the notion of  $V$ -bundle maps  $f: M \rightarrow N$  [10]. In case of a cover  $\{U\}$  for  $M$  and a cover  $\{W\}$  for  $N$  can be chosen such that each homomorphism  $\lambda: G_U \rightarrow G_W$  is injective, then  $f$  is called a proper  $V$ -bundle map [8, page 77].

Using the differentials of the elements in  $G_U$  we may construct the tangent  $V$ -bundle  $TX$ , which is a proper vector  $V$ -bundle over  $X$ .

A  $V$ -manifold is called orientable if all the elements of  $G_U$  and the overlap maps preserve the orientations.