

The Euler Class for Connective ko-Theory and an Application to Immersions of Quaternionic Projective Space

DONALD M. DAVIS & MARK MAHOWALD

1. Introduction. If an $8k$ -plane bundle θ over X admits a Spin structure, its ko-Euler class $\chi_0(\theta) \in \text{ko}^{8k}(X)$ is the class of the composite

$$X \xrightarrow{i} T(\theta) \xrightarrow{g} M\text{Spin}_{8k} \xrightarrow{a_0} bO_{8k},$$

where $T(\theta)$ is the Thom space of θ , i the usual inclusion, g the ‘‘Thomification’’ of the map $X \rightarrow B\text{Spin}_{8k}$ which classifies θ . $bO_{8k} = B\langle 8k \rangle$ is the $8k^{\text{th}}$ space in the spectrum for connective ko-theory, and a_0 the Atiyah-Bott-Shapiro orientation ([4]). If ξ_s denotes the Hopf bundle over RP^s , and the above bundle θ admits $(s + 1)$ linearly independent sections, then $\theta \otimes \xi_s$ admits a nonzero section over $X \times RP^s$, so that $\chi_0(\theta \otimes \xi_s) = 0$. ([5]). If θ is a symplectic bundle, then $\chi_0(\theta \otimes \xi_s) = \sum (-1)^i e_i(\theta) \otimes B^{2k-i}$, where $e_i(\theta) \in \text{ko}^{4i}(X)$ is the symplectic Pontryagin class and $B \in \text{ko}^4(RP)$ is a canonical generator. This result is straightforward to prove (see Theorem 2.1) and known ([6]), and yields the geometric dimension results of [8] when applied to multiples of the Hopf bundle over quaternionic projective space QP^n . There are similar results for the ku-Euler class $\chi_u(\)$ of complex vector bundles, which is in fact defined for Spin^c -bundles. However, for Spin bundles which are not symplectic, there does not seem to be a useful formula for $\chi_0(\theta \otimes \xi)$.

The stable normal bundle η of QP^n is not a complex bundle and hence not symplectic; it is a Spin bundle and hence a Spin^c -bundle. We calculate $\chi_0(\eta \otimes \xi)$ by finding a stable complex bundle θ over CP^{2n} such that $j^*\eta = \theta$ as Spin^c -bundles, for then $c_j^*(\chi_0(\eta \otimes \xi)) = \chi_u(\theta \otimes \xi)$, where c denotes complexification. We find

$$(1.1) \quad \chi_0(\eta \otimes \xi) = B^{L+n}(2Z - B)(Z - B)^{-(n+1)} \in \text{ko}^{4L}(QP^n \times RP^{4L-4n+d-1})$$

where $Z \in \text{ko}^4(QP^n)$ is the canonical generator. If (1.1) is nonzero, then QP^n cannot be immersed in R^{8n-d} and from this we immediately obtain all known non-immersions ([15, 13]) of QP^n .

Let $\alpha(n)$ denote the number of 1’s in the binary expansion of n .