

# Combinatorial Embeddings of Surfaces as Complex Algebraic Curves

R. EPHRAIM & R. S. KULKARNI

**§0. Introduction.** It is well known that a homology class  $d$  in  $H_2(\mathbf{P}_2(\mathbf{C}), \mathbf{Z}) \approx \mathbf{Z}$  where  $\mathbf{P}_2(\mathbf{C})$  denotes the complex projective space of dimension two can be represented by a combinatorially embedded 2-sphere with a single cone-like singularity (cf. [K-M, §2]). Interestingly such a sphere can also be represented by a complex algebraic curve whose equation in homogeneous coordinates is given by  $yz^{d-1} = x^d$ . This suggests the problem: *which closed orientable surfaces can be combinatorially embedded in  $\mathbf{P}_2(\mathbf{C})$  as complex algebraic curves of a given degree  $d$ ?* This problem in a weaker form without the reference to the degree  $d$  (in which case the solution turns out to be easy) was brought to the attention of the second author by T. Banchoff.

Let  $\Sigma_g$  denote a closed orientable surface of genus  $g$ . If  $g = (d - 1)(d - 2)/2$  then  $\Sigma_g$  can be realized by a *nonsingular* algebraic curve of degree  $d$  in  $\mathbf{P}_2(\mathbf{C})$ . In general due to the Plücker formulas (cf. §1, (1.1)),  $g$  must satisfy:  $g \leq (d - 1)(d - 2)/2$ . The problem whether  $\Sigma_g$  can be realized by a plane algebraic curve then amounts to whether an algebraic curve of geometric genus  $g$  can be found all of whose singularities are analytically irreducible. We can show that in certain “stable” ranges namely when  $g$  is close either to 0 or to  $(d - 1)(d - 2)/2$  there exist algebraic curves of degree  $d$  realizing  $\Sigma_g$ . Moreover, certain multiples of numbers less than or equal to  $d$  can also be possible genreses.

**Theorem 2.1.** *Let  $d$  be the smallest positive integer satisfying  $g \leq (d - 1)(d - 2)/2$ . Then  $\Sigma_g$  can be realized by a complex algebraic curve of degree  $d$  in  $\mathbf{P}_2(\mathbf{C})$  with at most one analytically irreducible singular point of multiplicity 2.*

**Corollary 3.6.** *For a fixed  $g$  and all sufficiently large  $d$  (e.g.  $\geq 2g + 1$ ) such that 6 does not divide  $d$ ,  $\Sigma_g$  can be realized by a complex algebraic curve of degree  $d$  in  $\mathbf{P}_2(\mathbf{C})$ .*

The proofs are constructive. In particular Corollary 3.6 is a consequence of essentially complete classification of “Kummer curves” (cf. §3) which represent combinatorially embedded surfaces.