

Extension Theorems for BMO

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§1. Introduction. Let $\mathcal{D} \subseteq \mathbf{R}^n$, $n \geq 2$ be a connected open set. We say that a function φ is in $\text{BMO}(\mathcal{D})$ if φ is locally integrable and

$$(1.1) \quad \sup_Q \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| dx = \|\varphi\|_{*,\mathcal{D}} < \infty$$

The above supremum is taken over all cubes $Q \subset \mathcal{D}$, and φ_Q denotes the mean value of φ over Q ,

$$\varphi_Q = \frac{1}{|Q|} \int_Q \varphi dx.$$

Under the norm $\|\cdot\|_{*,\mathcal{D}}$, $\text{BMO}(\mathcal{D})$ is a Banach space of functions modulo constants. An equivalent norm on $\text{BMO}(\mathcal{D})$ is defined if in (1.1) we only take the supremum over all cubes $Q \subset \mathcal{D}$ with sides parallel to the axes. This fact we will not prove here; its proof should become clear after reading Section 2 of this paper. We adopt the conventions $\text{BMO}(\mathbf{R}^n) = \text{BMO}$ and $\|\cdot\|_{*,\mathbf{R}^n} = \|\cdot\|_*$. See [12] and [18] for elementary properties of BMO functions.

The first problem treated in this paper is to find necessary and sufficient conditions on \mathcal{D} so that every function in $\text{BMO}(\mathcal{D})$ be the restriction to \mathcal{D} of a BMO function.

Let $\{Q_k\}$ be the dyadic Whitney decomposition of \mathcal{D} . Then $\mathcal{D} = \cup Q_k$ and

$$(1.2) \quad Q_j^0 \cap Q_k^0 = \emptyset, \quad j \neq k$$

$$(1.3) \quad 1 \leq \frac{d(Q_j, \mathcal{D}^c)}{\text{side}(Q_j)} \leq 4\sqrt{n}, \quad j = 1, 2, \dots$$

$$(1.4) \quad \frac{1}{4} \leq \frac{\text{side}(Q_j)}{\text{side}(Q_k)} \leq 4 \quad \text{if } Q_j \cap Q_k \neq \emptyset.$$

Here $d(\cdot, \cdot)$ denotes Euclidean distance. See [20, page 167] for details.

In Section 2 we will define two distance functions, d_1 and d_2 , on the set of Whitney cubes, $E = \{Q_j\}$. Our first theorem can then be formulated as follows: