

# *Analytic Properties of the Boundary of the Numerical Range*

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**1. Introduction.** Let  $\mathcal{H}$  be a complex, separable Hilbert space with inner product  $(\cdot, \cdot)$ , and norm  $\|\cdot\|$ ;  $B(\mathcal{H})$  denotes the set of bounded linear operators on  $\mathcal{H}$ . For  $T \in B(\mathcal{H})$ , the numerical range, its closure, and its topological boundary are, in order:  $W(T)$ ,  $W(T)^-$ , and  $\partial W(T)$ ;  $\sigma(T)$  is the spectrum.

The main result, obtained in Section 3 and precisely stated in Corollary 3.6, is that the boundary of the numerical range of a compact operator consists of a countable number of analytic arcs; singularities can accumulate at 0 or the endpoints of certain line segments. If 0 is not on the boundary, only a finite number of arcs are required. The self-contained proof used here relies chiefly on a theorem of Rellich [5, page 57]; B. Sz.-Nagy has an elegant, somewhat more elementary proof: see [4, page 376] concerning the behavior of isolated eigenvalues of a self-adjoint operator undergoing analytic perturbation. Agler [1] claims to have a proof of the main result.

As a byproduct of the techniques used to prove the main result, the results of Lancaster [2] concerning “corners” of the numerical range of a compact operator are obtained in a self-contained, elementary way. These techniques do not, however, extend to his more general results.

The rest of the paper is organized as follows: in Section 2, necessary geometric quantities are introduced and briefly discussed; in Section 3, the proof of the main result is given; in Section 4, the “corner” results mentioned above are obtained; in Section 5, boundary behavior is summarized, closing remarks are made and an example of a certain type of singularity is displayed.

**2. Geometric quantities.** Let  $W$  be a closed, bounded, convex subset of  $\mathbb{C}$  with 0 as an interior point. For every unit vector  $e^{i\theta}$  making an angle  $\theta$  with the positive real axis, there is precisely one line,  $L_\theta$ , tangent to  $W$  for which  $e^{i\theta}$  is both normal and outward drawn from  $W$ . The distance from  $L_\theta$  to 0 will be denoted by  $a(\theta)$ . (See Figure 1.)

No proofs will be given for the following propositions:

**Proposition 2.1.**  $w \in L_\theta$  if and only if  $\operatorname{Re}(e^{-i\theta} w) = a(\theta)$ .

**Proposition 2.2.** Every boundary point is on at least one tangent line; every tangent line intersects the boundary in at least one point.