

A Wiener-Lévy Theorem for Quotients, with Applications to Volterra Equations

KENNETH B. HANNSGEN

1. Introduction. The use of transform methods to solve a linear integral or integrodifferential equation leads to the problem of determining properties of the solution from properties of its transform. For Volterra equations of convolution type on \mathbf{R}^+ ($= [0, \infty)$) the complex Fourier transform, given by

$$(1.1) \quad A(\zeta) = a^*(\zeta) = \int_0^\infty e^{i\zeta t} a(t) dt \quad (\zeta \in Z),$$

($Z = \{\text{Im } \zeta \geq 0\}$) is often appropriate. If u is the solution of an equation, or if u is the difference between the solution and an approximate solution, the set on which u^* converges is not known at first; but if $u \in L^1(\mathbf{R}^+)$, then u^* is continuous in Z and analytic in the interior Z^0 , and $u^*(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$ ($\zeta \in Z$). In this paper we give conditions which ensure that a given function $f(\zeta)$ is the transform of some u in $L^1(\mathbf{R}^+)$.

Among other things, we shall assume that

$$(1.2) \quad f(\zeta) = \frac{\varphi(\mathbf{A}(\zeta), \zeta)}{\psi(\mathbf{A}(\zeta), \zeta)},$$

where $\mathbf{A} = (A_1, A_2, \dots, A_N)$ and each A_n is the transform of a function a_n in a certain subclass of $LL^1(\mathbf{R}^+)$ (LL^1 means locally integrable). φ and ψ are analytic in their $N + 1$ variables, but $\varphi(\mathbf{A}(\zeta), \zeta)$ and $\psi(\mathbf{A}(\zeta), \zeta)$ vanish at the nonzero real points $\tau = \omega_j$, $1 \leq j \leq J < \infty$. Thus the present results complement a recent theorem of D. F. Shea and S. Wainger [14] (extended by G. S. Jordan and R. L. Wheeler [8]), in which $\psi \equiv 1$ and $N = 1$. If, further, φ is independent of ζ and $a_1 \in L^1(\mathbf{R}^+)$, the matter reduces to one form of the Wiener-Lévy Theorem [13, page 63].

We discuss applications to Volterra equations in Section 2.

We shall consider functions a in $LL^1(\mathbf{R}^+)$ which satisfy hypotheses $H(M, D)$ as follows (M and D denote nonnegative integers).

$$H(M, 0): \quad \int_0^\infty t^M |a(t)| dt < \infty$$

$$H(M, 1): \quad a \text{ is a nonnegative and nonincreasing, and } \int_0^\infty t^{M-1} a(t) dt < \infty.$$